Last updated: November 1, 2026

🗺️ Four-year roadmap · Year 2, Spring

About this course

This free course has 10 units and 6 labs. Those are the real counts, not a longer syllabus cut short.

A check is one number or code result on a course page that a script recomputes before the page is published. The printed number and the recomputed number have to match.

> This is a free self-study course outline written by Saving Planets. It is not a university course, it carries no academic credit, and no school is involved.
>
> Its scope follows a one-semester sophomore linear algebra course. It is modeled on the topics and depth of MIT 18.06 Linear Algebra taught by Gilbert Strang (free on MIT OpenCourseWare), and on the Georgia Tech and Purdue sophomore linear algebra courses, using the free Georgia Tech text Interactive Linear Algebra by Dan Margalit and Joseph Rabinoff and Jim Hefferon's free Linear Algebra, which we link where it helps. None of those authors or schools endorses this page.
>
> What we add: every unit ties to real planet work with a cited public source (GPS, Landsat 9, NASA sea-level satellites, NOAA weather supercomputers, EPA input–output and recycling data, USGS building studies, NASA's EMIT mineral spectrometer), every number on the page is re-computed by a checking script before it goes live, and every lab runs in a free spreadsheet or on paper.
>
> Work through it in order. Each unit has goals, key ideas, equations, step-by-step examples, practice with full answers, common mistakes, flashcards and a quiz. Your progress is saved only in this browser.

How this course helps save the planet

Linear algebra is the language of data about the planet. A satellite pixel is a vector, a GPS fix is the solution of a linear system, a weather forecast is millions of equations solved again and again, and an economy's pollution can be traced with one matrix.

The same few ideas, elimination, projection, eigenvalues and the SVD, turn raw measurements into maps of forests, sea-level trends and mineral dust. Each unit ends with a short, sourced section. Values that are not from a source are labeled as sample values.

For instructors

You can teach this course, or let a student work through it alone. It is free. Saving Planets does not give a grade or school credit.

- This page is the whole course: 10 units and 6 labs (MATH 230).
- Plan on 15 weeks, about 6 to 9 hours a week, including the lab weeks.
- Follow the 15-week schedule. Read the unit, do the practice on paper, then open the answer. In a lab week, do the lab or the home version.
- A check is one number or code result on the page that a script recomputes before we publish. The number you see and the recomputed number have to match.
- Use Print this page, or download the PDF study guide. The formula sheet is near the end.
- Ask: how does this unit help people and the planet? Each unit has a green box with a sourced fact.
- Copy the teacher guide template and write in your class name, the time you have, and the questions you will ask. Open the teacher guide template

Before you start

Prerequisites

- High-school algebra: solving two equations in two unknowns, and working with functions.
- Calculus I helps with limits and derivatives in a few examples, but no calculus is needed for most of the course.
- Comfort with a spreadsheet; Intro to Programming helps for the optional Python versions of the labs.
- Calculus III and Differential Equations pair well with this course but are not required.

How to study

- Do every computation by hand first on small (2 × 2 and 3 × 3) examples, then check it with a spreadsheet or calculator.
- Think of Ax two ways: as dot products of the rows with x, and as a combination of the columns of A.
- Check every answer: plug x back into Ax = b, multiply A by its inverse, or test Av = λv.
- Keep track of sizes: an m × n matrix times an n × p matrix gives m × p, and sizes that do not match mean an error.
- Draw pictures in 2D and 3D: lines, planes, projections and stretches make the rules easy to remember.
- Do every practice problem on paper before opening the answer, and retake the quizzes a few days later.

What you need

- A computer with free spreadsheet software (its MMULT, MINVERSE, MDETERM and LINEST functions do matrix work).
- Graph paper, a ruler, a protractor and a pencil.
- Optional: free Python with NumPy for the larger versions of the labs.
- A string and push pins or tape for the trilateration lab, and a printed map or sheet of paper.
- A grayscale photo or a small grid of numbers for the image-compression lab.

Suggested 15-week schedule

About 6 to 9 hours a week: reading, practice problems and, in lab weeks, the lab.

Week	Unit	Topics	Reading
1	Vectors: Linear Combinations, Lengths and Dot Products	Vectors, linear combinations, span, length	MIT 18.06 L1; ILA 2.1–2.2
2	Vectors: Linear Combinations, Lengths and Dot Products	Dot products, angles, orthogonality; Lab A; start elimination	ILA 6.1
3	Systems of Linear Equations and Gaussian Elimination	Elimination, echelon forms, solution sets; Lab B	MIT 18.06 L2; ILA 1.1–1.3
4	Matrix Algebra: Multiplication, Inverses and LU Factorization	Matrix multiplication, inverses, A = LU; Lab C	MIT 18.06 L3–4; ILA 3.4–3.5
5	Linear Models: Input–Output (Leontief) Economics, Networks and Balancing	Input–output models, networks, balancing	Hefferon Topics; EPA USEEIO
6	Vector Spaces, Subspaces, Independence, Basis, Rank and the Four Fundamental Subspaces	Subspaces, column space, null space, Ax = 0	MIT 18.06 L5–7; ILA 2.6
7	Vector Spaces, Subspaces, Independence, Basis, Rank and the Four Fundamental Subspaces	Independence, basis, dimension, rank, four subspaces; midterm practice	MIT 18.06 L8–10; ILA 2.5, 2.7, 2.9
8	Orthogonality, Projections, Least Squares and Gram–Schmidt	Orthogonality and projections	MIT 18.06 L14–15; ILA 6.3
9	Orthogonality, Projections, Least Squares and Gram–Schmidt	Least squares, Gram–Schmidt, A = QR; Lab D	MIT 18.06 L16–17; ILA 6.4–6.5
10	Determinants: Properties, Cofactors, Cramer’s Rule and Volume	Determinants, cofactors, Cramer’s rule, volume	MIT 18.06 L18–20; ILA 4.1–4.3

Week	Unit	Topics	Reading
11	Eigenvalues and Eigenvectors	Eigenvalues, eigenvectors, power iteration, modes	MIT 18.06 L21; ILA 5.1–5.2, 5.5
12	Diagonalization, Matrix Powers, Markov Chains and Linear Dynamical Systems	Diagonalization, Markov chains, dynamical systems; Lab E	MIT 18.06 L22–25; ILA 5.4, 5.6
13	The Singular Value Decomposition, Principal Component Analysis and Positive Definite Matrices	Positive definite matrices and the SVD	MIT 18.06 L27–29
14	The Singular Value Decomposition, Principal Component Analysis and Positive Definite Matrices	Low-rank approximation, PCA, condition number; Lab F	MIT 18.06 L29; Strang book site
15	Review	Review and final practice set	All units

Your progress

Tick units and labs as you finish them, and take the quizzes. Progress and quiz scores are saved only in this browser, never sent to us.

Units

Vectors: Linear Combinations, Lengths and Dot Products

Vectors in two, three and n dimensions, adding and scaling them, linear combinations and span, length, unit vectors, the dot product, the angle between vectors, orthogonality and the Cauchy–Schwarz and triangle inequalities.

Learning objectives

By the end of this unit you can:

- Add, scale and combine vectors by hand and in pictures.
- Compute lengths, unit vectors and dot products.
- Find the angle between two vectors and test for orthogonality.
- Describe the span of one, two or three vectors.

Key concepts

- A vector in $\mathbb{R}^n$ is an ordered list of n numbers. In 2D and 3D it is an arrow; in higher dimensions it is a list, such as the 11 band values of one satellite pixel.
- A linear combination $c_1v_1 + c_2v_2 + \dots$ scales vectors and adds them. The span is the set of all linear combinations: a line, a plane or all of space.
- Length $\|v\| = \sqrt{(v \cdot v)}$. Dividing by the length gives a unit vector $u = v/\|v\|$ with length 1.
- Dot product $v \cdot w = v_1w_1 + v_2w_2 + \dots = \|v\|\|w\|\cos\theta$. If $v \cdot w = 0$ the vectors are orthogonal (perpendicular).
- Cauchy–Schwarz: $|v \cdot w| \le \|v\|\|w\|$. Triangle inequality: $\|v + w\| \le \|v\| + \|w\|$.

Key equations

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

\|v\|=_1^2+v_2^2+·s+v_n^2

Length

$$v \cdot w = v_1w_1 + \cdots + v_nw_n$$

v· w=v_1w_1+·s+v_nw_n

Dot product

$$\cos\theta = \frac{v \cdot w}{\|v\|\,\|w\|}$$

θ=(v· w)/(\|v\| \|w\|)

Angle between vectors

$$|v \cdot w| \le \|v\| \, \|w\|$$

Cauchy–Schwarz inequality

Worked examples, step by step

Example 1.1 Find the length of v = (3, 4) and the unit vector in its direction.

1. ‖v‖ = √(3² + 4²) = √25 = 5.
2. u = v/5.

Answer: ‖v‖ = 5, u = (0.6, 0.8).

Example 1.2 Find the angle between v = (3, 4) and w = (1, 2).

1. v·w = 3 + 8 = 11.
2. ‖v‖ = 5, ‖w‖ = √5 = 2.236.
3. cos θ = 11/(5 × 2.236) = 0.9839.

Answer: θ = 10.30°.

Example 1.3 Write 2(1, 0, 1) + 3(0, 1, 1) as one vector. Is (2, 3, 4) in the span of these two vectors?

1. 2(1, 0, 1) + 3(0, 1, 1) = (2, 3, 5).
2. Any combination c_1(1, 0, 1) + c_2(0, 1, 1) = (c_1, c_2, c_1 + c_2), so the third entry must equal the sum of the first two.

Answer: (2, 3, 5); (2, 3, 4) is not in the span because 2 + 3 ≠ 4.

Practice problems

Work each on paper first, then check.

1. Find ‖(1, 2, 2)‖.

   √(1 + 4 + 4) = 3.

2. Find (1, 2, 3)·(4, −5, 6).

4 − 10 + 18 = 12.

3. Are (2, −1) and (1, 2) orthogonal?

   Yes: 2 − 2 = 0.

4. Find the unit vector in the direction of (6, 8).

   ‖(6, 8)‖ = 10, so (0.6, 0.8).

Common misconceptions

Myth: The dot product is a vector.

Actually: It is a single number (a scalar).

Myth: ‖v + w‖ = ‖v‖ + ‖w‖ always.

Actually: Only when v and w point the same way; otherwise it is smaller.

Myth: Vectors only make sense in 2D or 3D.

Actually: Data vectors can have hundreds of entries; the same rules apply.

Flashcards

Linear combination

$c_1v_1 + c_2v_2 + \dots$

Span

All linear combinations of a set of vectors.

Length

$\|v\| = \sqrt{(v \cdot v)}$.

Dot product

$v \cdot w = \Sigma\, v_i w_i = \|v\|\|w\|\cos\theta$.

Orthogonal

$v \cdot w = 0$.

Unit vector

$v/\|v\|$.

Self-check quiz

1. ‖(3, 4)‖ equals…

7

5

25

12

5. $\sqrt{(9 + 16)} = 5$.

2. v·w = 0 means the vectors are...

parallel

orthogonal

equal

unit vectors

orthogonal. $\cos \theta = 0$, so $\theta = 90°$.

3. The span of two non-parallel vectors in $\mathbb{R}^3$ is...

a point

a line

a plane

all of $\mathbb{R}^3$

a plane. Two independent directions fill a plane.

4. Doubling v changes ‖v‖ by a factor of...

1

2

4

$\sqrt{2}$

2. $\|cv\| = |c|\|v\|$.

Try this quiz again

How this helps save the planet

Every Landsat 9 pixel is a vector. According to USGS, Landsat 9's Operational Land Imager 2 records 9 spectral bands at 30 m (15 m for the panchromatic band) and its Thermal Infrared Sensor 2 records 2 thermal bands at 100 m, so each spot on the ground becomes a list of 11 numbers. The data are 14-bit, 2^{14} = 16,384 shades per band, and Landsat 9 collects about 750 scenes a day, all free.

Forest and crop health is read from the red band (band 4, 0.64–0.67 μm) and the near-infrared band (band 5, 0.85–0.88 μm). The vegetation index NDVI = (NIR − red)/(NIR + red) compares two entries of the pixel vector. With sample reflectances of 0.40 (NIR) and 0.08 (red), NDVI = 0.32/0.48 = 0.667, typical of healthy vegetation; bare soil gives values near zero.

On Saving Planets: Earth Observation · Deforestation

Sources: USGS: Landsat 9 (https://www.usgs.gov/landsat-missions/landsat-9) · Interactive Linear Algebra, 2.1 Vectors (https://textbooks.math.gatech.edu/ila/vectors.html)

Free learning resources

- Interactive Linear Algebra, 2.1 Vectors (https://textbooks.math.gatech.edu/ila/vectors.html): Free text with interactive pictures.
- Interactive Linear Algebra, 6.1 Dot products and orthogonality (https://textbooks.math.gatech.edu/ila/dot-product.html): Lengths, angles and orthogonality.
- MIT OpenCourseWare 18.06 Linear Algebra (Strang, Spring 2010) (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/): Lecture 1: the geometry of linear equations.
- PhET: Vector Addition (https://phet.colorado.edu/en/simulations/vector-addition): Interactive.

☐ **I finished this unit**

Systems of Linear Equations and Gaussian Elimination

Linear systems in row and column pictures, augmented matrices, elimination to echelon form, back substitution, reduced row echelon form, pivots and free variables, and when a system has one, none or infinitely many solutions.

Learning objectives

By the end of this unit you can:

- Write a system as Ax = b and as an augmented matrix.
- Solve systems by elimination and back substitution.
- Find the reduced row echelon form and identify pivots and free variables.
- Decide whether a system has one, no or infinitely many solutions.

Key concepts

- Row picture: each equation is a line (2D) or plane (3D) and the solution is where they meet. Column picture: b is a combination of the columns of A, with weights x.
- Three row operations do not change the solutions: swap rows, multiply a row by a nonzero number, and add a multiple of one row to another.
- Elimination produces an upper-triangular (echelon) form; back substitution then solves from the bottom up. Pivots are the first nonzero entries of each row.
- A column without a pivot gives a free variable and infinitely many solutions. A row 0 = c with c ≠ 0 means no solution (inconsistent).
- Squared-distance equations like (x − a)² + (y − b)² = d² are not linear, but subtracting one from another cancels x² and y² and leaves linear equations. That is how positioning problems become linear systems.

Key equations

$$A\mathbf{x} = \mathbf{b}$$

Matrix form

$$R_i \leftarrow R_i - \ell_{ij} R_j, \;\; \ell_{ij} = a_{ij}/a_{jj}$$

```
R_i R_i-_ijR_j,\ \ _ij=a_ij/a_jj
```

Elimination step and multiplier

$$\text{solutions: one, none, or infinitely many}$$

The only three cases

$$d = c\,t$$

```
d=c t
```

Range from signal travel time (GPS)

Worked examples, step by step

Example 2.1 Solve x + y + z = 6, 2x + y − z = 1, x − y + 2z = 5.

1. $R_2 - 2R_1$: −y − 3z = −11. $R_3 - R_1$: −2y + z = −1.
2. $R_3 - 2R_2$: 7z = 21, so z = 3.
3. Back substitute: −y − 9 = −11 gives y = 2; then x = 6 − 2 − 3 = 1.

Answer: (x, y, z) = (1, 2, 3).

Example 2.2 Beacons sit at (0, 0), (10, 0) and (0, 10) km. A receiver measures squared distances 25, 65 and 45 km². Find its position.

1. Equation 1 minus equation 2: $x^2 + y^2 - [(x-10)^2 + y^2] = 25 - 65$, so 20x − 100 = −40.
2. Equation 1 minus equation 3: 20y − 100 = −20.
3. Solve the two linear equations.

Answer: (x, y) = (3, 4) km.

Example 2.3 How many solutions do x + 2y = 3, 2x + 4y = 6 have? And x + 2y = 3, 2x + 4y = 7?

1. $R_2 - 2R_1$ gives 0 = 0 in the first system: one pivot, one free variable.
2. In the second, $R_2 - 2R_1$ gives 0 = 1.

Answer: Infinitely many; none.

Practice problems

Work each on paper first, then check.

1. Solve x + y = 5, x − y = 1.

   Adding gives 2x = 6: x = 3, y = 2.

2. A GPS-style signal takes 0.00505783 s to arrive. How far did it travel at 299,792,458 m/s?

d = ct = 1,516,299 m, about 1,516 km.

3. Find the reduced row echelon form of [[1, 2, 3], [2, 5, 8]] (augmented).

   $R_2 - 2R_1 = [0, 1, 2]$; $R_1 - 2R_2 = [1, 0, -1]$: x = −1, y = 2.

4. A 3 × 5 system has 3 pivots. How many free variables?

   5 − 3 = 2.

Common misconceptions

Myth: Multiplying a row by 0 is a legal row operation.
Actually: It destroys information; the multiplier must be nonzero.

Myth: A system with more unknowns than equations has no solution.
Actually: If consistent, it has infinitely many; it can also be inconsistent.

Myth: A zero row means no solution.
Actually: Only a row 0 = c with c ≠ 0 does; 0 = 0 just means a free variable.

Flashcards

Augmented matrix

[A | b].

Pivot

First nonzero entry in a row of echelon form.

Free variable

Variable whose column has no pivot.

RREF

Pivots 1, zeros above and below.

Inconsistent

A row 0 = c, c ≠ 0.

Back substitution

Solve the triangular system from the bottom up.

Self-check quiz

1. A row 0 0 0 | 5 in echelon form means...

one solution

no solution

infinitely many

x = 5

no solution. 0 = 5 is impossible.

2. Which is NOT a row operation?

swap two rows

add a multiple of a row to another

multiply a row by 3

square a row

square a row. Squaring is not linear.

3. A pivot-free column gives...

a free variable

no solution

a zero determinant only

a unique solution

a free variable. Its variable can be chosen freely.

4. In the column picture, Ax = b asks whether b is...

orthogonal to A

a combination of the columns of A

a row of A

an eigenvector

a combination of the columns of A. $Ax = x_1a_1 + x_2a_2 + \dots$

Try this quiz again

How this helps save the planet

GPS is a linear algebra machine. GPS.gov reports that at least 24 operational satellites are kept available 95% of the time and that 31 operational satellites have been flown for more than a decade, at about 20,200 km altitude, each circling Earth twice a day. They sit in six equally spaced orbital planes so that users can view at least four satellites from virtually any point on the planet.

GPS.gov's trilateration activity explains the idea: distance = rate × time, with the rate equal to the speed of light, 299,792,458 m/s, and distances from at least 3 satellites fix a 3D position. A real receiver's clock is also unknown, so in the standard textbook model the unknowns are x, y, z and the clock offset, four unknowns that need four satellites; differencing the range equations, as in the worked example, turns them into a linear system. A clock error of just one microsecond would put a range off by 299.8 m. Precise positions guide farm tractors that apply less fertilizer, ships on efficient routes and the survey stations that measure sea level and ice.

On Saving Planets: Earth Observation · Space

Sources: GPS.gov: Space segment (https://www.gps.gov/space-segment) · GPS.gov: Trilateration (NOAA classroom activity) (https://www.gps.gov/trilateration)

Free learning resources

- Interactive Linear Algebra, 1.1 Systems of linear equations (https://textbooks.math.gatech.edu/ila/systems-of-eqns.html): Free text section.
- Interactive Linear Algebra, 1.2 Row reduction (https://textbooks.math.gatech.edu/ila/row-reduction.html): Echelon forms step by step.
- MIT OpenCourseWare 18.06 Linear Algebra (Strang, Spring 2010) (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/): Lecture 2: elimination with matrices.
- Jim Hefferon, Linear Algebra (free text) (https://hefferon.net/linearalgebra/): Chapter One: linear systems.

Watch the lecture

Lecture 1: The geometry of linear equations (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/resources/lecture-1-the-geometry-of-linear-equations/) (in English)

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☐ I finished this unit

Matrix Algebra: Multiplication, Inverses and LU Factorization

Matrix–vector and matrix–matrix products, the rules (and non-rules) of matrix algebra, transposes, elimination matrices, the inverse and Gauss–Jordan, the 2 × 2 inverse formula, A = LU, and the cost of elimination.

Learning objectives

By the end of this unit you can:

- Multiply matrices and check that sizes match.
- Find inverses by the 2 × 2 formula and by Gauss–Jordan.
- Factor A = LU and use it to solve systems.
- Estimate the cost of elimination as n grows.

Key concepts

- $(AB)_{ij}$ = row i of A dotted with column j of B. An m × n times an n × p matrix is m × p. In general AB ≠ BA.
- The inverse A^{-1} satisfies $AA^{-1} = A^{-1}A = I$. It exists exactly when A has n pivots (det A ≠ 0). $(AB)^{-1} = B^{-1}A^{-1}$ and $(AB)^{T} = B^{T}A^{T}$.
- Gauss–Jordan: row-reduce [A | I] to [I | A^{-1}].
- Elimination without row swaps records its multipliers in a lower-triangular L with ones on the diagonal, giving A = LU. Solve Lc = b, then Ux = c; with L and U saved, each new right side is cheap.
- Elimination on an n × n matrix costs about $n^3/3$ multiply–add operations; solving with saved L and U costs about n^2 per right side.

Key equations

$$(AB)_{ij} = \sum_{k} a_{ik} b_{kj}$$

```
(AB)_ij=_k a_ikb_kj
```

Matrix product

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

```
&b\&d^-1=(1)/(ad-bc)&-b\\-c&a
```

2 × 2 inverse

$$A = LU$$

A=LU

LU factorization

$$\text{cost} \approx n^3/3$$

Elimination operation count

Worked examples, step by step

Example 3.1 Find AB and BA for A = [[1, 2], [3, 4]] and B = [[0, 1], [1, 0]].

1. AB: row 1 · columns = (2, 1); row 2 = (4, 3).
2. BA swaps the rows of A: [[3, 4], [1, 2]].

Answer: AB = [[2, 1], [4, 3]], BA = [[3, 4], [1, 2]], so AB ≠ BA.

Example 3.2 Invert A = [[4, 7], [2, 6]].

1. ad − bc = 24 − 14 = 10.
2. A^{-1} = (1/10)[[6, −7], [−2, 4]].

Answer: A^{-1} = [[0.6, −0.7], [−0.2, 0.4]].

Example 3.3 Factor A = [[2, 1], [6, 8]] as LU.

1. Multiplier ℓ_{21} = 6/2 = 3.
2. $R_2 - 3R_1$ = [0, 5].

Answer: L = [[1, 0], [3, 1]], U = [[2, 1], [0, 5]].

Practice problems

Work each on paper first, then check.

1. What size is a 3 × 2 matrix times a 2 × 4 matrix?

   3 × 4.

2. Invert [[2, 1], [5, 3]].

   det = 1, inverse [[3, −1], [−5, 2]].

3. About how many operations does elimination take for n = 1000?

   $n^3/3 \approx 3.33 \times 10^8$.

4. If A and B are invertible, what is $(AB)^{-1}$?

   $B^{-1}A^{-1}$.

Common misconceptions

Myth: AB = BA.

Actually: Matrix multiplication is usually not commutative.

Myth: $(AB)^{-1} = A^{-1}B^{-1}$.

Actually: The order reverses: $B^{-1}A^{-1}$.

Myth: Computers solve Ax = b by computing A^{-1}.

Actually: They factor A = LU (or similar); forming A^{-1} is slower and less accurate.

Flashcards

Identity

AI = IA = A.

Inverse

$AA^{-1} = I$.

2 × 2 inverse

Swap a, d; negate b, c; divide by ad − bc.

Transpose rule

$(AB)^T = B^TA^T$.

LU

A = LU from elimination.

Cost

About $n^3/3$ for elimination.

Self-check quiz

1. A 2 × 3 matrix times a 3 × 5 matrix is...

2 × 5

3 × 3

5 × 2

undefined

2 × 5. Outer sizes.

2. A matrix is invertible exactly when it has...

no zeros

n pivots

a zero row

equal entries

n pivots. Full set of pivots.

3. Doubling n multiplies elimination cost by about...

2

4

8

16

8. n^3 grows $2^3 = 8$ times.

4. L in A = LU holds...

the pivots

the multipliers

the inverse

the eigenvalues

the multipliers. Below-diagonal multipliers, ones on the diagonal.

Try this quiz again

How this helps save the planet

Weather and climate forecasts are, at their core, enormous linear-algebra jobs repeated many times a day. NOAA reported in August 2023 that its twin operational supercomputers, Dogwood in Manassas, Virginia and Cactus in Phoenix, Arizona, were upgraded by 20% to 14.5 petaflops each, so that together the forecast system can process 29 quadrillion calculations per second, and that with its research supercomputers its total capacity is 49 petaflops.

The $n^3/3$ rule shows why speed matters. As a sample, a dense system with n = 100,000 unknowns needs about 3.33×10^{14} operations; at a sustained 14.5×10^{15} operations per second that is about 0.023 s, but doubling n multiplies the work by 8. Real models use sparse matrices and clever solvers, yet the same cubic growth is what NOAA says the extra power will pay for: higher-resolution models that improve forecasts and warnings.

On Saving Planets: Intro to Programming · Solutions

Sources: NOAA: NOAA completes upgrade to weather and climate supercomputer system (Aug. 10, 2023) (https://www.noaa.gov/news-release/noaa-completes-upgrade-to-weather-and-climate-supercomputer-system) · MIT OpenCourseWare 18.06 Linear Algebra (Strang, Spring 2010) (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/)

Free learning resources

- Interactive Linear Algebra, 3.4 Matrix multiplication (https://textbooks.math.gatech.edu/ila/matrix-multiplication.html): Free text section.
- Interactive Linear Algebra, 3.5 Matrix inverses (https://textbooks.math.gatech.edu/ila/matrix-inverses.html): Inverses and Gauss–Jordan.
- MIT OpenCourseWare 18.06 Linear Algebra (Strang, Spring 2010) (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/): Lectures 3–4: multiplication, inverses and A = LU.
- MIT OpenCourseWare 18.06SC Linear Algebra (self-paced, Fall 2011) (https://ocw.mit.edu/courses/18-06sc-linear-algebra-fall-2011/): Recitation videos on LU.

Watch the lecture

Lecture 3: Multiplication and inverse matrices (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/resources/lecture-3-multiplication-and-inverse-matrices/) (in English)

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☐ **I finished this unit**

Linear Models: Input–Output (Leontief) Economics, Networks and Balancing

Building linear models from real systems: the Leontief input–output model x = Ax + d and its solution x = (I − A)$^{-1}$d, environmentally extended models that turn spending into emissions, network flows and balancing chemical equations.

Learning objectives

By the end of this unit you can:

- Set up and solve a Leontief input–output model.
- Interpret the entries of (I − A)$^{-1}$.
- Extend an input–output model with emission intensities to find a footprint.
- Write flow-balance and chemical-balance problems as linear systems.

Key concepts

- In an input–output model, column j of the consumption matrix A lists what sector j buys from each sector to make one dollar of output.
- Total output x must cover what industries use (Ax) plus final demand d: x = Ax + d, so (I − A)x = d and x = (I − A)$^{-1}$d.
- Entry (i, j) of the Leontief inverse (I − A)$^{-1}$ is the total output of sector i needed, directly and through all supply chains, for one dollar of final demand from sector j.
- Environmentally extended models multiply output by a row of emission intensities e (for example kg CO_2 per dollar): footprint = e·x.
- Conservation laws give linear equations too: flow in equals flow out at every node of a network, and atoms of each element balance in a chemical reaction.

Key equations

$$\mathbf{x} = A\mathbf{x} + \mathbf{d}$$

Input–output balance

$$\mathbf{x} = (I - A)^{-1}\mathbf{d}$$

```
=(I-A)^-1
```

Leontief solution

$$\text{footprint} = \mathbf{e} \cdot \mathbf{x}$$

`footprint=·`

Environmentally extended model

$$\sum \text{in} = \sum \text{out}$$

`in= out`

Flow balance at a node

Worked examples, step by step

Example 4.1 A two-sector economy has A = [[0.2, 0.3], [0.1, 0.4]] and final demand d = (100, 50) (sample values, $ million). Find the total output.

1. I − A = [[0.8, −0.3], [−0.1, 0.6]], det = 0.48 − 0.03 = 0.45.
2. $(I - A)^{-1}$ = (1/0.45)[[0.6, 0.3], [0.1, 0.8]].
3. x = (1/0.45)(60 + 15, 10 + 40).

Answer: x = (166.7, 111.1) $ million.

Example 4.2 With sample emission intensities e = (0.5, 2.0) kg CO_2 per dollar, find the footprint of that final demand.

1. footprint = 0.5 × 166.7 + 2.0 × 111.1 (in kg per $ × $ million).

Answer: 305.6 million kg = 305.6 kt CO_2 (sample).

Example 4.3 Balance x CH_4 + y O_2 → z CO_2 + w H_2O.

1. C: x = z. H: 4x = 2w. O: 2y = 2z + w.
2. Set x = 1: z = 1, w = 2, y = 2.

Answer: CH_4 + 2 O_2 → CO_2 + 2 H_2O.

Practice problems

Work each on paper first, then check.

1. In the worked model, how much output of sector 1 does $1 of final demand for sector 2 require in total?

Entry (1, 2) of $(I - A)^{-1} = 0.3/0.45 = 0.667$.

2. A single-sector economy uses 0.25 of its own output. What total output meets a demand of 300?

   $x = 300/(1 - 0.25) = 400$.

3. A node has inflows 40 and 25 and one outflow f_1 plus an outflow of 30. Find f_1.

   $f_1 = 65 - 30 = 35$.

4. Balance $H_2 + O_2 \to H_2O$.

   $2\ H_2 + O_2 \to 2\ H_2O$.

Common misconceptions

Myth: Total output equals final demand.
Actually: Industries also consume each other's output, so x is larger than d.

Myth: The footprint comes only from the sector you buy from.
Actually: The Leontief inverse adds every upstream supplier.

Myth: Any A gives a sensible economy.
Actually: Column sums must be below 1 for $(I - A)^{-1}$ to have nonnegative entries.

Flashcards

Consumption matrix

Inputs per dollar of output.

Leontief model

$x = Ax + d$.

Leontief inverse

$(I - A)^{-1}$, total requirements.

Extended model

footprint = e·x.

Flow balance

In = out at each node.

Homogeneous system

$Ax = 0$.

Self-check quiz

1. The Leontief solution is...

$x = Ad$

$x = (I - A)^{-1}d$

$x = A^{-1}d$

$x = d - A$

$x = (I - A)^{-1}d$. From $x = Ax + d$.

2. Column j of A describes...

what sector j sells

what sector j buys per dollar of output

final demand

emissions

what sector j buys per dollar of output. Inputs per unit of output.

3. A footprint in an extended model is...

$e \cdot x$

$e + x$

det A

x/e

$e \cdot x$. Intensity times output, summed.

4. Balancing a chemical equation is a...

nonlinear problem

homogeneous linear system

determinant

eigenvalue problem

homogeneous linear system. Atom counts give Ax = 0.

Try this quiz again

How this helps save the planet

The U.S. EPA uses exactly this model to trace pollution through the economy. According to EPA, its USEEIO models combine data on economic transactions between 389 industry sectors with environmental data for those sectors, covering land, water, energy and mineral use, air pollution, nutrients and toxics, to evaluate the impacts of goods and services. EPA describes them as useful for carbon footprinting, and they support its sustainable materials management work.

A 389-sector consumption matrix has $389^2 = 151{,}321$ entries, far too many for hand work but easy for the elimination and LU methods of Unit 3. The Leontief inverse is what lets a city, company or household see the emissions hidden in its supply chains, not just at the factory gate.

On Saving Planets: Solutions · General Chemistry I

Sources: U.S. EPA: US Environmentally-Extended Input-Output (USEEIO) models (https://www.epa.gov/land-research/us-environmentally-extended-input-output-useeio-models) · Interactive Linear Algebra, 1.1 Systems of linear equations (https://textbooks.math.gatech.edu/ila/systems-of-eqns.html)

Free learning resources

- U.S. EPA: US Environmentally-Extended Input-Output (USEEIO) models (https://www.epa.gov/land-research/us-environmentally-extended-input-output-useeio-models): EPA's environmentally extended input–output models.
- Jim Hefferon, Linear Algebra (free text) (https://hefferon.net/linearalgebra/): Topic: Leontief input–output analysis.
- MIT OpenCourseWare 18.06 Linear Algebra (Strang, Spring 2010) (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/): Applications lectures and problem sets.
- Interactive Linear Algebra, 1.2 Row reduction (https://textbooks.math.gatech.edu/ila/row-reduction.html): Solving the systems.

☐ **I finished this unit**

Vector Spaces, Subspaces, Independence, Basis, Rank and the Four Fundamental Subspaces

Vector spaces and subspaces, the column space and null space, solving Ax = 0 and Ax = b completely, linear independence, basis and dimension, rank, the rank–nullity theorem and Strang's four fundamental subspaces.

Learning objectives

By the end of this unit you can:

- Test whether a set is a subspace.
- Find bases for the column space and null space of a matrix.
- Test vectors for linear independence.
- Use rank–nullity and the dimensions of the four fundamental subspaces.

Key concepts

- A subspace contains the zero vector and is closed under addition and scalar multiplication. Lines and planes through the origin are subspaces; ones that miss the origin are not.
- Column space C(A): all Ax, the set of b for which Ax = b is solvable. Null space N(A): all x with Ax = 0.
- Vectors are linearly independent if the only combination giving zero is the all-zero one. A basis is an independent spanning set; every basis of a space has the same number of vectors, its dimension.
- Rank r = number of pivots = dim C(A). Rank–nullity: r + dim N(A) = n, the number of columns.
- Four fundamental subspaces of an m × n matrix of rank r: C(A) (dim r) and N(A^T) (dim m − r) in $\mathbb{R}^m$; C(A^T) (dim r) and N(A) (dim n − r) in $\mathbb{R}^n$. N(A) is orthogonal to the row space.

Key equations

$$C(A) = \{A\mathbf{x}\}, \;\; N(A) = \{\mathbf{x} : A\mathbf{x} = 0\}$$

`C(A)=\,\ \ N(A)=\:A=0\`

Column space and null space

$$r + \dim N(A) = n$$

`r+ N(A)=n`

Rank–nullity

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$$

Complete solution of Ax = b

$$\dim N(A^T) = m - r$$

N(A^T)=m-r

Left null space

Worked examples, step by step

Example 5.1 Find the rank and a null-space basis of A = [[1, 2, 3], [2, 4, 6], [1, 0, 1]].

1. $R_2 - 2R_1 = 0$; $R_3 - R_1 = [0, -2, -2]$. Two pivots, so r = 2.
2. Ax = 0: x + z = 0 and −2y − 2z = 0, with z free.
3. Take z = 1: x = −1, y = −1.

Answer: r = 2, N(A) spanned by (−1, −1, 1); dim N(A) = 3 − 2 = 1.

Example 5.2 Are (1, 2, 3), (4, 5, 6), (7, 8, 9) independent?

1. Row-reduce the matrix with these columns: $R_2 - 2R_1$ and $R_3 - 3R_1$ give rows [0, −3, −6] and [0, −6, −12], which are multiples.
2. Only two pivots.

Answer: No: (7, 8, 9) = 2(4, 5, 6) − (1, 2, 3).

Example 5.3 A 5 × 8 matrix has rank 3. Give the dimensions of its four fundamental subspaces.

1. C(A) and $C(A^T)$: r = 3.
2. N(A): n − r = 8 − 3; $N(A^T)$: m − r = 5 − 3.

Answer: 3, 3, 5 and 2.

Practice problems

Work each on paper first, then check.

1. Is the line y = 2x + 1 a subspace of $\mathbb{R}^2$?

   No: it does not contain (0, 0).

2. A 4 × 4 matrix has a 2-dimensional null space. What is its rank?

4 − 2 = 2.

3. What is the dimension of the space of 2 × 2 matrices?

   4.

4. Can 4 vectors in $\mathbb{R}^3$ be independent?

   No: at most 3.

Common misconceptions

Myth: Any plane in $\mathbb{R}^3$ is a subspace.

Actually: Only planes through the origin.

Myth: Rank counts nonzero entries.

Actually: It counts pivots (independent columns).

Myth: The null space lives in the same space as b.

Actually: N(A) is in $\mathbb{R}^n$ (the x side); b is in $\mathbb{R}^m$.

Flashcards

Subspace

Contains 0; closed under + and scaling.

Column space

All Ax.

Null space

All x with Ax = 0.

Basis

Independent spanning set.

Rank

Number of pivots.

Rank–nullity

r + dim N(A) = n.

Self-check quiz

1. Rank–nullity says r + dim N(A) equals...

m

n

m + n

r

n. Number of columns.

2. Ax = b is solvable exactly when b is in...

N(A)

C(A)

$N(A^T)$

$C(A^T)$

C(A). b must be a combination of the columns.

3. The null space is orthogonal to...

the column space

the row space

b

the left null space

the row space. Each row dotted with x is 0.

4. A basis of $\mathbb{R}^3$ has how many vectors?

2

3

4

any number

3. Dimension 3.

Try this quiz again

How this helps save the planet

Dimension counts unknowns, and better forecasts need more of them. NOAA said its 2023 supercomputer upgrade would let the U.S. Global Forecast System improve its horizontal resolution from 13 km to 9 km, helping smaller-scale features be seen and simulated.

A grid with spacing h has about $(1/h)^2$ columns per unit area, so going from 13 km to 9 km multiplies the number of grid columns by $(13/9)^2 = 2.09$. Every one of those columns carries temperature, wind, humidity and pressure at many levels, so the state vector of a forecast lives in a space with a vastly higher dimension, and the matrices that update it grow to match.

On Saving Planets: Calculus III · Differential Equations

Sources: NOAA: NOAA completes upgrade to weather and climate supercomputer system (Aug. 10, 2023) (https://www.noaa.gov/news-release/noaa-completes-upgrade-to-weather-and-climate-supercomputer-system) · Interactive Linear Algebra, 2.7 Basis and dimension (https://textbooks.math.gatech.edu/ila/dimension.html)

Free learning resources

- Interactive Linear Algebra, 2.6 Subspaces (https://textbooks.math.gatech.edu/ila/subspaces.html): Free text section.
- Interactive Linear Algebra, 2.5 Linear independence (https://textbooks.math.gatech.edu/ila/linear-independence.html): Independence tests.
- Interactive Linear Algebra, 2.9 The rank theorem (https://textbooks.math.gatech.edu/ila/rank-thm.html): Rank and nullity.
- MIT OpenCourseWare 18.06 Linear Algebra (Strang, Spring 2010) (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/): Lectures 5–10: spaces, rank and the four subspaces.

Watch the lecture

Lecture 10: The four fundamental subspaces (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/resources/lecture-10-the-four-fundamental-subspaces/) (in English)

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☐ I finished this unit

Orthogonality, Projections, Least Squares and Gram–Schmidt

Orthogonal vectors and subspaces, projection onto a line and onto a subspace, the normal equations $A^TA\hat{x} = A^Tb$, fitting lines and curves to data by least squares, orthonormal bases, Gram–Schmidt and A = QR.

Learning objectives

By the end of this unit you can:

- Project a vector onto a line and onto a column space.
- Fit a line to data with the normal equations.
- Build an orthonormal basis with Gram–Schmidt.
- Explain why least squares is a projection.

Key concepts

- The projection of b onto the line through a is p = (a·b/a·a) a; the error e = b − p is orthogonal to a.
- When Ax = b has no solution (more equations than unknowns, noisy data), choose $\hat{x}$ to make $\|b - A\hat{x}\|$ as small as possible. The error must be orthogonal to C(A), giving the normal equations $A^TA\hat{x} = A^Tb$.
- For a line y = C + Dt through points (t_i, y_i), the columns of A are all ones and the t_i values.
- Orthonormal vectors q_i satisfy $q_i \cdot q_j = 0$ $(i \neq j)$ and $\|q_i\| = 1$. With Q having orthonormal columns, $Q^TQ = I$ and projections become $p = QQ^Tb$.
- Gram–Schmidt subtracts from each new vector its projections on the earlier ones, then normalizes. The result is A = QR with R upper triangular.

Key equations

$$\mathbf{p} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a}$$

Projection onto a line

$$A^T A\hat{\mathbf{x}} = A^T \mathbf{b}$$

A^TA=A^T

Normal equations

$$P = A(A^T A)^{-1} A^T$$

`P=A(A^TA)^-1A^T`

Projection matrix

$$A = QR$$

`A=QR`

Gram–Schmidt factorization

Worked examples, step by step

Example 6.1 Project b = (2, 3) onto a = (1, 1).

1. a·b = 5, a·a = 2.
2. p = 2.5(1, 1); e = b − p = (−0.5, 0.5), and e·a = 0.

Answer: p = (2.5, 2.5).

Example 6.2 Fit y = C + Dt to (0, 1), (1, 2), (2, 4) by least squares.

1. A has rows (1, 0), (1, 1), (1, 2); A^TA = [[3, 3], [3, 5]], A^Tb = (7, 10).
2. det = 6; C = (5 × 7 − 3 × 10)/6, D = (3 × 10 − 3 × 7)/6.

Answer: C = 0.833, D = 1.5: y = 0.833 + 1.5t.

Example 6.3 Apply Gram–Schmidt to a_1 = (1, 1, 0), a_2 = (1, 0, 1).

1. q_1 = (1, 1, 0)/√2.
2. $a_2 - (a_2 \cdot q_1)q_1$ = (1, 0, 1) − ½(1, 1, 0) = (0.5, −0.5, 1), length √1.5 = 1.225.

Answer: q_1 = (0.707, 0.707, 0), q_2 = (0.408, −0.408, 0.816).

Practice problems

Work each on paper first, then check.

1. Project (3, 4) onto (1, 0).

   (3, 0).

2. Write the normal equations for fitting y = Dt (no intercept) to (1, 2), (2, 3).

$\Sigma t^2\, D = \Sigma ty$: $5D = 8$, $D = 1.6$.

3. Is $Q = [[0, 1], [1, 0]]$ orthogonal?

   Yes: $Q^TQ = I$.

4. If b is already in $C(A)$, what is the least-squares error?

   Zero: $p = b$.

Common misconceptions

Myth: Least squares solves $Ax = b$ exactly.

Actually: It finds the closest $A\hat{x}$ when no exact solution exists.

Myth: Any basis makes projections easy.

Actually: Orthonormal bases do; otherwise you need $(A^TA)^{-1}$.

Myth: A straight-line fit means the data are linear.

Actually: Check the residuals; a curve may fit better (add a t^2 column).

Flashcards

Projection on a line

$p = (a \cdot b / a \cdot a)a$.

Normal equations

$A^TA\hat{x} = A^Tb$.

Projection matrix

$P = A(A^TA)^{-1}A^T$.

Orthonormal

$q_i \cdot q_j = 0$, $\|q_i\| = 1$.

Gram–Schmidt

Subtract projections, normalize.

QR

$A = QR$.

Self-check quiz

1. The normal equations are…

$Ax = b$

$A^TA\hat{x} = A^Tb$

$AA^Tx = b$

$A^Tx = b$

$A^TA\hat{x} = A^Tb$. Error orthogonal to C(A).

2. The least-squares error vector is orthogonal to...

b

the column space

the null space

$\hat{x}$

the column space. That is what makes it shortest.

3. For Q with orthonormal columns, Q^TQ =...

0

Q

I

QQ^T

I. Dot products 0 or 1.

4. Gram–Schmidt produces...

$A = LU$

$A = QR$

$A = PDP^{-1}$

$A = U\Sigma V^T$

A = QR. Orthonormal Q, triangular R.

Try this quiz again

How this helps save the planet

Sea-level rise is measured by fitting curves to noisy satellite data. NASA reports that satellite altimeters, TOPEX/Poseidon, Jason-1, -2 and -3 and now Sentinel-6 Michael Freilich, have tracked global mean sea level since 1993, that it has risen 114.4 mm since then, and that the rate of rise has increased from 0.08 inches per year (0.20 cm per year) in 1993 to 0.17 inches per year (0.44 cm per year) now, about 2.2 times faster.

A straight-line least-squares fit gives an average rate; adding a t^2 column to A fits a curve whose changing slope measures the acceleration NASA describes. Lab D fits sample data shaped like the NASA record with exactly the normal equations from this unit.

On Saving Planets: Oceans · Earth Observation

Sources: NASA: Global mean sea level (Vital Signs) (https://sealevel.nasa.gov/vital-signs/global-mean-sea-level/) · Interactive Linear Algebra, 6.5 The method of least squares (https://textbooks.math.gatech.edu/ila/least-squares.html)

Free learning resources

- Interactive Linear Algebra, 6.3 Orthogonal projection (https://textbooks.math.gatech.edu/ila/projections.html): Free text section.
- Interactive Linear Algebra, 6.5 The method of least squares (https://textbooks.math.gatech.edu/ila/least-squares.html): Least squares with pictures.
- Interactive Linear Algebra, 6.4 Orthogonal sets (https://textbooks.math.gatech.edu/ila/orthogonal-sets.html): Gram–Schmidt.
- MIT OpenCourseWare 18.06 Linear Algebra (Strang, Spring 2010) (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/): Lectures 14–17: orthogonality, projections, least squares, Gram–Schmidt.

Watch the lecture

Lecture 16: Projection matrices and least squares (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/resources/lecture-16-projection-matrices-and-least-squares/) (in English)

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☐ I finished this unit

Determinants: Properties, Cofactors, Cramer’s Rule and Volume

The determinant from its properties, computing it by elimination and by cofactor expansion, det(AB) = det A det B, the test for invertibility, Cramer’s rule, and the determinant as the area or volume scale factor of a linear map.

Learning objectives

By the end of this unit you can:

- Compute 2 × 2 and 3 × 3 determinants by cofactors and by elimination.
- Use the properties of determinants under row operations.
- Solve small systems by Cramer’s rule.
- Interpret |det A| as an area or volume scale factor.

Key concepts

- det A ≠ 0 exactly when A is invertible. The determinant of a triangular matrix is the product of its diagonal entries, so det A = ± (product of pivots).
- Row operations: a swap flips the sign, scaling a row by c scales det by c, and adding a multiple of one row to another leaves det unchanged.
- det(AB) = det A · det B, det(A^T) = det A and det(A^{-1}) = 1/det A.
- Cofactor expansion along a row: det A = Σ $a_{ij}C_{ij}$ with $C_{ij} = (-1)^{i+j}$ times the minor. Cramer’s rule: x_i = det(B_i)/det A, where B_i has b in column i.
- A linear map x ↦ Ax multiplies areas (2D) or volumes (3D) by |det A|; a negative determinant also flips orientation.

Key equations

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

```
&b\&d=ad-bc
```

2 × 2 determinant

$$\det(AB) = \det A \, \det B$$

```
(AB)= A B
```

Product rule

$$x_i = \frac{\det B_i}{\det A}$$

`x_i=( B_i)/( A)`

Cramer’s rule

$$\operatorname{area}(AS) = |\det A| \operatorname{area}(S)$$

`area(AS)=| A| area(S)`

Area scale factor

Worked examples, step by step

Example 7.1 Find det [[1, 2, 0], [0, 1, 3], [2, 0, 1]] by cofactors along row 1.

1. 1 × (1 × 1 − 3 × 0) = 1.
2. − 2 × (0 × 1 − 3 × 2) = −2 × (−6) = 12.
3. + 0 × (...) = 0.

Answer: det = 13.

Example 7.2 Solve 2x + y = 5, x + 3y = 10 by Cramer’s rule.

1. det A = 6 − 1 = 5.
2. x = det[[5, 1], [10, 3]]/5 = (15 − 10)/5.
3. y = det[[2, 5], [1, 10]]/5 = (20 − 5)/5.

Answer: x = 1, y = 3.

Example 7.3 Find the area of the parallelogram with sides (3, 0) and (1, 2).

1. Area = |det [[3, 1], [0, 2]]| = |6 − 0|.

Answer: 6 square units.

Practice problems

Work each on paper first, then check.

1. det [[3, 1], [2, 4]]?

12 − 2 = 10.

2. det [[2, 0, 1], [1, 3, 2], [1, 1, 1]]?

   2(3 − 2) − 0 + 1(1 − 3) = 0, so it is singular.

3. If det A = 4 for a 3 × 3 A, what is det(2A)?

   $2^3 \times 4 = 32$.

4. If det A = 5, what is $\det(A^{-1})$?

   1/5 = 0.2.

Common misconceptions

Myth: det(A + B) = det A + det B.

Actually: False in general; only det(AB) splits.

Myth: det(2A) = 2 det A.

Actually: For n × n, $\det(2A) = 2^n \det A$.

Myth: Cofactor expansion is the fast way.

Actually: It takes about n! steps; elimination takes about $n^3/3$.

Flashcards

2 × 2 det

ad − bc.

Invertibility

det A ≠ 0.

Product rule

det(AB) = det A det B.

Row swap

Changes the sign.

Cramer’s rule

$x_i = \det B_i / \det A$.

Area factor

|det A|.

Self-check quiz

1. Swapping two rows multiplies det by...

0

1

−1

2

−1. Sign flips.

2. det A = 0 means A is...

orthogonal

singular

symmetric

the identity

singular. Not invertible.

3. det of a triangular matrix is...

the trace

the product of the diagonal

zero

the sum of the diagonal

the product of the diagonal. Expand down the diagonal.

4. A map with det = 3 changes areas by a factor of...

1/3

3

9

√3

3. |det A|.

Try this quiz again

How this helps save the planet

Turning pixels into land area is a determinant. USGS lists Landsat 9's multispectral bands at 30 m and its panchromatic band at 15 m. The map from pixel indices (i, j) to ground coordinates is, locally, a matrix; for an aligned 30 m grid it is diag(30, 30) m, whose determinant is 900, so each pixel covers 900 m². The 15 m panchromatic grid has determinant 225 m², so four panchromatic pixels fit in one multispectral pixel.

Counting pixels classified as forest and multiplying by this scale factor is how satellite records measure forest loss in hectares (1 ha = 10,000 m², about 11.1 Landsat pixels). When an image is rotated or sheared to fit a map, the determinant of the new matrix keeps the area right.

On Saving Planets: Deforestation · Earth Observation

Sources: USGS: Landsat 9 (https://www.usgs.gov/landsat-missions/landsat-9) · Interactive Linear Algebra, 4.3 Determinants and volumes (https://textbooks.math.gatech.edu/ila/determinants-volumes.html)

Free learning resources

- Interactive Linear Algebra, 4.1 Determinants: definition (https://textbooks.math.gatech.edu/ila/determinants-definitions-properties.html): Free text section.
- Interactive Linear Algebra, 4.3 Determinants and volumes (https://textbooks.math.gatech.edu/ila/determinants-volumes.html): Geometric meaning.
- MIT OpenCourseWare 18.06 Linear Algebra (Strang, Spring 2010) (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/): Lectures 18–20: determinants, cofactors, Cramer's rule.
- Jim Hefferon, Linear Algebra (free text) (https://hefferon.net/linearalgebra/): Chapter Four: determinants.

Watch the lecture

Lecture 20: Cramer's rule, inverse matrix, and volume (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/resources/lecture-20-cramers-rule-inverse-matrix-and-volume/) (in English)

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☐ **I finished this unit**

Eigenvalues and Eigenvectors

Eigenvectors as directions a matrix only stretches, the characteristic equation det(A − λI) = 0, trace and determinant checks, eigenvalues of special matrices, complex eigenvalues and rotation, power iteration, and natural modes of vibrating structures.

Learning objectives

By the end of this unit you can:

- Find eigenvalues from det(A − λI) = 0 and eigenvectors from the null space of A − λI.
- Check eigenvalues with the trace and determinant.
- Recognize real, repeated and complex eigenvalues.
- Use power iteration to find the dominant eigenvalue.

Key concepts

- Av = λv with v ≠ 0: the eigenvector v keeps its direction and is scaled by the eigenvalue λ.
- The eigenvalues are the roots of det(A − λI) = 0, a polynomial of degree n. For each λ, the eigenvectors fill the null space of A − λI.
- Checks: the eigenvalues sum to the trace (sum of the diagonal) and multiply to det A.
- Symmetric matrices have real eigenvalues and orthogonal eigenvectors. Rotation matrices have complex eigenvalues. Triangular matrices show their eigenvalues on the diagonal.
- Power iteration: repeatedly multiply a starting vector by A and rescale; it lines up with the eigenvector of the largest |λ|. For vibration, Kv = ω²Mv gives the natural frequencies ω and mode shapes v.

Key equations

$$A\mathbf{v} = \lambda\mathbf{v}$$

Eigenvalue equation

$$\det(A - \lambda I) = 0$$

```
(A- I)=0
```

Characteristic equation

$$\lambda_1 + \cdots + \lambda_n = \operatorname{tr} A, \;\; \lambda_1 \cdots \lambda_n = \det A$$

```
_1+·s+_n=,\ \ _1·s_n= A
```

Trace and determinant checks

$$K\mathbf{v} = \omega^2 M\mathbf{v}$$

K=ω^2M

Natural modes of a structure

Worked examples, step by step

Example 8.1 Find the eigenvalues and eigenvectors of A = [[2, 1], [1, 2]].

1. det(A − λI) = (2 − λ)² − 1 = λ² − 4λ + 3 = 0.
2. λ = 3: A − 3I = [[−1, 1], [1, −1]] gives v = (1, 1). λ = 1: v = (1, −1).
3. Check: 3 + 1 = 4 = trace; 3 × 1 = 3 = det.

Answer: λ = 3 with (1, 1); λ = 1 with (1, −1).

Example 8.2 Find the eigenvalues of [[4, 1], [2, 3]].

1. λ² − 7λ + 10 = 0 (trace 7, det 10).
2. (λ − 5)(λ − 2) = 0.

Answer: λ = 5 and λ = 2.

Example 8.3 A two-story sample building has equal floor masses m and story stiffness k, with K/m = (k/m)[[2, −1], [−1, 1]] and k/m = 100 s⁻². Find the fundamental frequency.

1. Eigenvalues of [[2, −1], [−1, 1]]: λ = (3 ± √5)/2 = 0.382 and 2.618.
2. ω_1^2 = 100 × 0.382 = 38.2 s⁻², ω_1 = 6.18 rad/s.
3. $f_1 = \omega_1/(2\pi)$, $T_1 = 1/f_1$.

Answer: f_1 = 0.984 Hz, period T_1 = 1.02 s (sample values).

Practice problems

Work each on paper first, then check.

1. Eigenvalues of [[5, 0], [0, −2]]?

   5 and −2 (diagonal).

2. A 3 × 3 matrix has eigenvalues 1, 2, 3. Find its trace and determinant.

Trace 6, det 6.

3. Eigenvalues of the 90° rotation [[0, −1], [1, 0]]?

   $\lambda^2 + 1 = 0$: $\lambda = \pm i$.

4. If $Av = 4v$, what is A^2v?

   16v.

Common misconceptions

Myth: The zero vector is an eigenvector.
Actually: Eigenvectors must be nonzero.

Myth: Row reduction keeps the eigenvalues.
Actually: It changes them; use $\det(A - \lambda I)$.

Myth: Every real matrix has real eigenvalues.
Actually: Rotations give complex ones; symmetric matrices are the safe case.

Flashcards

Eigenvector

$Av = \lambda v$, $v \neq 0$.

Characteristic equation

$\det(A - \lambda I) = 0$.

Trace check

$\Sigma\lambda$ = trace.

Determinant check

$\Pi\lambda = \det A$.

Power iteration

Repeated multiplication finds the top eigenvector.

Natural modes

$Kv = \omega^2 Mv$.

Self-check quiz

1. The eigenvalues of a triangular matrix are...

its pivots after swaps

its diagonal entries

all 1

its row sums

its diagonal entries. det(A − λI) is a product of diagonal terms.

2. The sum of the eigenvalues equals...

the determinant

the trace

the rank

n

the trace. Trace check.

3. Power iteration finds...

the smallest eigenvalue

the dominant eigenvalue

the determinant

the inverse

the dominant eigenvalue. Largest |λ| wins.

4. Symmetric matrices have...

complex eigenvalues

real eigenvalues and orthogonal eigenvectors

zero determinant

no eigenvectors

real eigenvalues and orthogonal eigenvectors. Spectral theorem.

Try this quiz again

How this helps save the planet

Earthquake engineers read buildings through their eigenvalues. A USGS study by Mehmet Çelebi (2022) used the fundamental periods and critical damping percentages of 41 tall buildings in several countries. It found a clear linear trend between building height and fundamental period for steel and reinforced-concrete buildings, though with a large scatter from differing structural designs, and that critical damping was clearly lower than 3%, consistent with other recent tall-building studies.

The fundamental period is set by the smallest eigenvalue of the stiffness–mass problem, as in the two-story sample example. Low damping means a tall building can ring for a long time when long-period earthquake waves match that period, which is why instrumented buildings and their measured modes guide safer, longer-lasting cities.

On Saving Planets: Differential Equations · Dynamics

Sources: USGS: Preliminary study of correlation of natural periods and damping percentages of tall buildings in several countries (Çelebi, 2022) (https://www.usgs.gov/publications/preliminary-study-correlation-natural-periods-and-damping-percentages-tall-buildings) · Interactive Linear Algebra, 5.1 Eigenvalues and eigenvectors (https://textbooks.math.gatech.edu/ila/eigenvectors.html)

Free learning resources

- Interactive Linear Algebra, 5.1 Eigenvalues and eigenvectors (https://textbooks.math.gatech.edu/ila/eigenvectors.html): Free text section.
- Interactive Linear Algebra, 5.2 The characteristic polynomial (https://textbooks.math.gatech.edu/ila/characteristic-polynomial.html): Finding eigenvalues.
- Interactive Linear Algebra, 5.5 Complex eigenvalues (https://textbooks.math.gatech.edu/ila/complex-eigenvalues.html): Rotation and scaling.
- MIT OpenCourseWare 18.06 Linear Algebra (Strang, Spring 2010) (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/): Lecture 21: eigenvalues and eigenvectors.

☐ **I finished this unit**

Diagonalization, Matrix Powers, Markov Chains and Linear Dynamical Systems

Diagonalizing $A = PDP^{-1}$, computing powers A^k quickly, symmetric matrices and the spectral theorem $A = Q\Lambda Q^T$, Markov (stochastic) matrices and steady states, and the long-run behavior of $x_{k+1} = Ax_k$.

Learning objectives

By the end of this unit you can:

- Diagonalize a matrix with n independent eigenvectors.
- Compute A^k and predict the long-run behavior of $x_{k+1} = Ax_k$.
- Find the steady state of a Markov chain.
- Use the spectral theorem for symmetric matrices.

Key concepts

- If A has n independent eigenvectors (the columns of P), then $A = PDP^{-1}$ with the eigenvalues on the diagonal of D, and $A^k = PD^kP^{-1}$.
- Write $x_0 = c_1v_1 + \dots + c_nv_n$. Then $x_k = A^kx_0 = c_1\lambda_1^kv_1 + \dots + c_n\lambda_n^kv_n$; the largest $|\lambda|$ dominates as k grows.
- A Markov matrix has nonnegative entries and columns that sum to 1. It always has $\lambda = 1$, and the other eigenvalues satisfy $|\lambda| \le 1$. The steady state is the eigenvector for $\lambda = 1$, scaled so its entries sum to 1.
- The second-largest $|\lambda|$ sets how fast a Markov chain forgets its start: the gap shrinks like $|\lambda_2|^k$.
- Spectral theorem: a real symmetric matrix can be written $A = Q\Lambda Q^T$ with orthonormal eigenvectors in Q.

Key equations

$$A = PDP^{-1}, \;\; A^k = PD^kP^{-1}$$

`A=PDP^-1,\ \ A^k=PD^kP^-1`

Diagonalization and powers

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_n\lambda_n^k\mathbf{v}_n$$

`_k=c_1_1^k_1+·s+c_n_n^k_n`

Solution of x_{k+1} = Ax_k

$$P\mathbf{q} = \mathbf{q}, \;\; \sum q_i = 1$$

P=,\ \ q_i=1

Markov steady state

$$A = Q\Lambda Q^T$$

A=Q Q^T

Spectral theorem

Worked examples, step by step

Example 9.1 Use diagonalization to find A^{10} for A = [[2, 1], [1, 2]].

1. From Unit 8: λ = 3, 1 with v = (1, 1), (1, −1). P = [[1, 1], [1, −1]], P^{-1} = ½[[1, 1], [1, −1]].
2. A^{10} = P diag(3^{10}, 1) P^{-1}; each entry is (3^{10} ± 1)/2.

Answer: A^{10} = [[29525, 29524], [29524, 29525]].

Example 9.2 A Markov matrix is P = [[0.8, 0.3], [0.2, 0.7]] (sample values). Find the steady state.

1. (P − I)q = 0: $-0.2q_1 + 0.3q_2 = 0$, so $q_1 = 1.5q_2$.
2. $q_1 + q_2 = 1$ gives $q_2 = 0.4$.

Answer: q = (0.6, 0.4).

Example 9.3 Starting from x_0 = (1, 0), find x_2 for the same P, and the general x_k.

1. x_1 = (0.8, 0.2); x_2 = (0.64 + 0.06, 0.16 + 0.14).
2. Eigenvalues are 1 and 0.5 (trace 1.5, det 0.5), so x_k = (0.6, 0.4) + 0.5^k(0.4, −0.4).

Answer: x_2 = (0.70, 0.30); x_k → (0.6, 0.4).

Practice problems

Work each on paper first, then check.

1. Is [[0.5, 0.2], [0.5, 0.8]] a Markov matrix?

   Yes: nonnegative, columns sum to 1.

2. Find its steady state.

$0.5q_1 = 0.2q_2$, so $q = (2/7, 5/7) = (0.286, 0.714)$.

3. If $A = PDP^{-1}$ with $D = \text{diag}(0.9, 0.5)$, what happens to $A^k x$ for large k?

   It decays to zero, dominated by 0.9^k.

4. Can $[[1, 1], [0, 1]]$ be diagonalized?

   No: $\lambda = 1$ twice but only one independent eigenvector.

Common misconceptions

Myth: Every matrix can be diagonalized.

Actually: It needs n independent eigenvectors; [[1, 1], [0, 1]] has only one.

Myth: A Markov chain's steady state depends on the start.

Actually: For a regular chain it does not; every start converges to the same q.

Myth: A^k means raising each entry to the k-th power.

Actually: Only diagonal matrices work that way; that is why we diagonalize.

Flashcards

Diagonalization

$A = PDP^{-1}$.

Matrix power

$A^k = PD^kP^{-1}$.

Markov matrix

Nonnegative, columns sum to 1.

Steady state

$Pq = q$, entries sum to 1.

Spectral theorem

$A = Q\Lambda Q^T$.

Defective matrix

Too few eigenvectors to diagonalize.

Self-check quiz

1. Every Markov matrix has the eigenvalue...

0

1

-1

2

1. Columns sum to 1.

2. A^k for $A = PDP^{-1}$ is...

P^kDP^k

PD^kP^{-1}

D^kP

$kPDP^{-1}$

PD^kP^{-1}. Inner $P^{-1}P$ cancel.

3. The convergence speed of a Markov chain is set by...

$\lambda_1 = 1$

$|\lambda_2|$

the trace

n

$|\lambda_2|$. The gap shrinks like $|\lambda_2|^k$.

4. Symmetric matrices diagonalize as...

LU

$Q\Lambda Q^T$

QR

$U\Sigma V^T$ only

$Q\Lambda Q^T$. Spectral theorem.

Try this quiz again

How this helps save the planet

Where materials end up is a Markov chain. EPA reports that the United States generated 292.4 million tons of municipal solid waste in 2018, 4.9 pounds per person per day. About 69 million tons were recycled and 25 million tons composted, a combined 32.1% rate; about 35 million tons (11.8%) were burned with energy recovery; and more than 146 million tons (50%) were landfilled.

If each year's shares stayed fixed, a material would leave use for these destinations with these probabilities, and only the recycled share would return to the economy. Lab E builds a sample transition matrix with a 32.1% recovery share and shows, with matrix powers and the $\lambda = 1$ eigenvector, how raising recycling changes how long material stays in use and how much reaches landfill.

On Saving Planets: Solutions

Sources: U.S. EPA: National overview, facts and figures on materials, wastes and recycling (https://www.epa.gov/facts-and-figures-about-materials-waste-and-recycling/national-overview-facts-and-figures-materials) · Interactive Linear Algebra, 5.6 Stochastic matrices (https://textbooks.math.gatech.edu/ila/stochastic-matrices.html)

Free learning resources

- Interactive Linear Algebra, 5.4 Diagonalization (https://textbooks.math.gatech.edu/ila/diagonalization.html): Free text section.
- Interactive Linear Algebra, 5.6 Stochastic matrices (https://textbooks.math.gatech.edu/ila/stochastic-matrices.html): Markov chains and steady states.
- MIT OpenCourseWare 18.06 Linear Algebra (Strang, Spring 2010) (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/): Lectures 22–25: diagonalization, Markov matrices, symmetric matrices.
- MIT OpenCourseWare 18.06SC Linear Algebra (self-paced, Fall 2011) (https://ocw.mit.edu/courses/18-06sc-linear-algebra-fall-2011/): Recitations on Markov matrices.

Watch the lecture

Lecture 22: Diagonalization and powers of A (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/resources/lecture-22-diagonalization-and-powers-of-a/) (in English)

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☐ I finished this unit

The Singular Value Decomposition, Principal Component Analysis and Positive Definite Matrices

Positive definite matrices and A^TA, the SVD $A = U\Sigma V^T$, singular values and the geometry of a matrix as rotate–stretch–rotate, best low-rank approximation and image compression, the condition number, and principal component analysis of data.

Learning objectives

By the end of this unit you can:

- Test a symmetric matrix for positive definiteness.
- Compute singular values from the eigenvalues of A^TA.
- Build a rank-k approximation and measure what it keeps.
- Run a small principal component analysis.

Key concepts

- A symmetric matrix is positive definite when $x^TAx > 0$ for every $x \neq 0$; equivalently all eigenvalues (or all pivots) are positive. A^TA is always symmetric and positive semidefinite.
- Every m × n matrix has an SVD $A = U\Sigma V^T$: V and U have orthonormal columns, and Σ holds the singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$, the square roots of the eigenvalues of A^TA.
- $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \ldots$. Keeping the first k terms gives the best rank-k approximation (Eckart–Young); the error in the 2-norm is σ_{k+1}.
- The condition number σ_1/σ_n measures how much errors in b can be amplified when solving $Ax = b$.
- PCA: center the data (subtract column means), then the right singular vectors (eigenvectors of the covariance matrix) are the principal directions, and σ_i^2 measures the variance each explains.

Key equations

$$A = U\Sigma V^T$$

`A=U V^T`

Singular value decomposition

$$\sigma_i = \sqrt{\lambda_i(A^TA)}$$

`σ_i=_i(A^TA)`

Singular values

$$A_k = \sum_{i=1}^{k} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$$

`A_k=_i=1^kσ_i_i_i^T`

Best rank-k approximation

$$\kappa = \sigma_1 / \sigma_n$$

Condition number

Worked examples, step by step

Example 10.1 Find the singular values of A = [[3, 0], [4, 5]].

1. A^TA = [[25, 20], [20, 25]], eigenvalues 45 and 5.
2. $\sigma_1 = \sqrt{45}$, $\sigma_2 = \sqrt{5}$. Check: $\sigma_1\sigma_2 = 15 = |\det A|$.

Answer: $\sigma_1 = 6.708$, $\sigma_2 = 2.236$.

Example 10.2 What share of A^TA's "energy" $\Sigma\sigma^2$ does the rank-1 approximation of that A keep?

1. $\sigma_1^2/(\sigma_1^2 + \sigma_2^2) = 45/50$.

Answer: 90%.

Example 10.3 A 1000 × 1000 grayscale image is stored as a rank-50 SVD. How many numbers are stored, compared with the full image?

1. Each term needs one u (1000), one v (1000) and one σ: 2001 numbers.
2. 50 × 2001 = 100,050 versus 1,000,000.

Answer: 100,050 numbers, about 10.0% of the original.

Practice problems

Work each on paper first, then check.

1. Is [[2, 1], [1, 2]] positive definite?

Yes: eigenvalues 3 and 1 are positive.

2. Find the condition number of diag(10, 0.1).

   10/0.1 = 100.

3. Centered 2D data have covariance [[2, 1], [1, 2]]. What share of the variance does the first principal component explain?

   Eigenvalues 3 and 1: 3/4 = 75%, along (1, 1).

4. What are the singular values of an orthogonal matrix?

   All equal to 1.

Common misconceptions

Myth: Singular values are the eigenvalues of A.

Actually: They are square roots of the eigenvalues of A^TA, and are never negative.

Myth: Only square matrices have an SVD.

Actually: Every matrix does.

Myth: PCA works on raw data.

Actually: Center the data first, and scale variables measured in different units.

Flashcards

SVD

$A = U\Sigma V^T$.

Singular value

$\sigma_i = \sqrt{\lambda_i}(A^TA)$.

Positive definite

$x^TAx > 0$, all $\lambda > 0$.

Eckart–Young

Truncated SVD is the best rank-k approximation.

Condition number

σ_1/σ_n.

PCA

SVD of centered data.

Self-check quiz

1. Singular values are...

any real numbers

nonnegative

always integers

eigenvalues of A

nonnegative. Square roots of eigenvalues of $A^TA \geq 0$.

2. The best rank-k approximation keeps...

the first k rows

the k largest singular terms

k random columns

the k smallest σ

the k largest singular terms. Eckart–Young.

3. A large condition number means...

a well-behaved system

errors can be amplified

A is orthogonal

det A = 1

errors can be amplified. Near-singular.

4. In PCA, the first principal direction is...

the mean

the top right singular vector of the centered data

the null space

the first row

the top right singular vector of the centered data. Direction of most variance.

Try this quiz again

How this helps save the planet

NASA JPL's EMIT imaging spectrometer on the International Space Station maps the minerals in Earth's dust-source regions. JPL says it collects more than 100,000 spectra every second, which is 3.6×10^8 spectra an hour, and that its mission has been extended through at least 2026. Dust matters for climate: JPL notes that darker minerals absorb sunlight and warm the air, while lighter minerals reflect it and cool.

Each spectrum is a long vector of reflectances at many wavelengths, and neighboring wavelengths are strongly correlated. PCA and the SVD compress such data into a few components, separate signal from noise, and pick out the signatures of individual minerals, the same mathematics as the image-compression lab.

On Saving Planets: Earth Observation · Space

Sources: NASA JPL: EMIT, Earth Surface Mineral Dust Source Investigation (https://earth.jpl.nasa.gov/emit/) · MIT OpenCourseWare 18.06 Linear Algebra (Strang, Spring 2010) (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/)

Free learning resources

- MIT OpenCourseWare 18.06 Linear Algebra (Strang, Spring 2010) (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/): Lectures 27–29: positive definite matrices and the SVD.
- MIT OpenCourseWare 18.06SC Linear Algebra (self-paced, Fall 2011) (https://ocw.mit.edu/courses/18-06sc-linear-algebra-fall-2011/): Session on the singular value decomposition.
- Gilbert Strang, Introduction to Linear Algebra (book site) (https://math.mit.edu/~gs/linearalgebra/): Strang's notes on the SVD and data.
- Jim Hefferon, Linear Algebra (free text) (https://hefferon.net/linearalgebra/): Topics on orthogonal matrices and projections.

Watch the lecture

Lecture 25: Symmetric matrices and positive definiteness (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/resources/lecture-25-symmetric-matrices-and-positive-definiteness/) (in English)

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☐ **I finished this unit**

Problem Sets

These practice problems live inside each unit. Work on paper first, then open the answer.

1. Vectors: Linear Combinations, Lengths and Dot Products
   4 problems
2. Systems of Linear Equations and Gaussian Elimination
   4 problems
3. Matrix Algebra: Multiplication, Inverses and LU Factorization
   4 problems
4. Linear Models: Input–Output (Leontief) Economics, Networks and Balancing
   4 problems
5. Vector Spaces, Subspaces, Independence, Basis, Rank and the Four Fundamental Subspaces
   4 problems
6. Orthogonality, Projections, Least Squares and Gram–Schmidt
   4 problems
7. Determinants: Properties, Cofactors, Cramer's Rule and Volume
   4 problems
8. Eigenvalues and Eigenvectors
   4 problems
9. Diagonalization, Matrix Powers, Markov Chains and Linear Dynamical Systems
   4 problems
10. The Singular Value Decomposition, Principal Component Analysis and Positive Definite Matrices
    4 problems

The lab (Computation lab)

The lab is the hands-on part of the course. Here it is free and for practice only, with no credit.

Every lab runs in a free spreadsheet or on paper; Python versions are optional. Take regular breaks from the screen, and handle pins and scissors with care.

Safety reference: OSHA: Computer workstations eTool (https://www.osha.gov/etools/computer-workstations)

Lab A: Pixel vectors and a vegetation index

WEEK 2

Goal: Treat satellite pixels as vectors: compute lengths, angles between pixels and the NDVI, and see that vegetation, soil and water point in different directions.

$$\cos\theta = \frac{\mathbf{p}\cdot\mathbf{q}}{\|\mathbf{p}\|\|\mathbf{q}\|}, \quad \mathrm{NDVI} = \frac{\mathrm{NIR}-\mathrm{red}}{\mathrm{NIR}+\mathrm{red}}$$

θ=·\|\|\|\|, NDVI=-redNIR+red

Angle between pixels and NDVI

Procedure

1. In a spreadsheet, enter three sample pixels as rows of (green, red, NIR) reflectances: vegetation (0.10, 0.08, 0.40), soil (0.20, 0.25, 0.30), water (0.06, 0.04, 0.02).
2. Compute each pixel's length with SQRT(SUMSQ(...)) and each pair's dot product with SUMPRODUCT.
3. Find the angle between each pair with ACOS and DEGREES.
4. Compute NDVI for each pixel. Optional: download a free Landsat scene and repeat for real pixels.

Data table

Pixel	Green	Red	NIR	NDVI
Vegetation	0.10	0.08	0.40	0.667
Soil	0.20	0.25	0.30	0.091
Water	0.06	0.04	0.02	−0.333

Sample reflectances chosen to look like typical vegetation, bare soil and water.

Sample calculation

1. Vegetation NDVI = (0.40 − 0.08)/(0.40 + 0.08) = 0.667.
2. Soil NDVI = 0.05/0.55 = 0.091; water NDVI = −0.02/0.06 = −0.333.
3. Angle vegetation–soil: dot = 0.160, lengths 0.420 and 0.4387, cos θ = 0.8683, θ = 29.7°.

> **Safety**
>
> - Take a short screen break every half hour.

Home or virtual alternative

- The whole lab runs in a free spreadsheet.

PhET: Vector Addition (https://phet.colorado.edu/en/simulations/vector-addition) · USGS: Landsat 9 (https://www.usgs.gov/landsat-missions/landsat-9)

Planet link: USGS lists Landsat 9's red (band 4) and near-infrared (band 5) bands, and its data are free, so anyone can map vegetation this way. USGS: Landsat 9 (https://www.usgs.gov/landsat-missions/landsat-9)

☐ **I completed this lab (wet or virtual)**

Lab B: Trilateration, from strings to a linear system

WEEK 3

Goal: Find a position from distances, first with strings on a map as in the GPS.gov activity, then by turning the circle equations into a linear system and solving it by elimination.

$$2(a_2 - a_1)x + 2(b_2 - b_1)y = d_1^2 - d_2^2 + a_2^2 - a_1^2 + b_2^2 - b_1^2$$

`2(a_2-a_1)x+2(b_2-b_1)y=d_1^2-d_2^2+a_2^2-a_1^2+b_2^2-b_1^2`

Difference of two circle equations

Procedure

1. Print the GPS.gov trilateration map (without scaling) or mark three “satellites” on graph paper at (0, 0), (10, 0) and (0, 10) cm.
2. Convert signal times to distances with d = ct (c = 299,792,458 m/s); for the paper version use the sample distances below.
3. Draw circles with a string and pencil and mark where they cross.
4. Subtract the circle equations in pairs to get two linear equations, solve them by elimination in a spreadsheet, and compare with your drawing.

Data table

Beacon	Position (cm)	Distance (cm)	d²
1	(0, 0)	5.000	25
2	(10, 0)	8.062	65
3	(0, 10)	6.708	45

Sample distances for a receiver at (3, 4) cm. The GPS.gov activity uses four satellites on a map at 1:21,283,839.

Sample calculation

1. Eq. 1 − Eq. 2: 20x − 100 = −40, x = 3. Eq. 1 − Eq. 3: 20y − 100 = −20, y = 4.
2. GPS.gov time A = 0.00505783 s gives d = 1,516,299 m; at 1:21,283,839 that is 7.12 cm on the map.

> **Safety**
>
> - Handle push pins and scissors with care.

Home or virtual alternative

- The string-and-paper version is the home lab.

GPS.gov: Trilateration activity (https://www.gps.gov/trilateration) · Interactive Linear Algebra, 1.2 Row reduction (https://textbooks.math.gatech.edu/ila/row-reduction.html)

Planet link: GPS.gov designed this activity with NOAA; real receivers solve the same kind of system, with a clock unknown, many times a second. GPS.gov: Trilateration (NOAA classroom activity) (https://www.gps.gov/trilateration) · GPS.gov: Space segment (https://www.gps.gov/space-segment)

- [ ] **I completed this lab (wet or virtual)**

Lab C: Matrices as transformations of the plane

WEEK 4

Goal: See matrices act on shapes: stretch, shear, rotate and reflect a square on graph paper, and confirm that areas scale by |det A|.

$$\text{area}(AS) = |\det A| \, \text{area}(S)$$

```
area(AS)=| A| area(S)
```

Area scale factor

Procedure

1. Draw the unit square with corners (0, 0), (1, 0), (1, 1), (0, 1).
2. Multiply each corner by A = [[2, 1], [0, 1]] (a shear and stretch), by R = [[0, −1], [1, 0]] (a 90° rotation) and by F = [[1, 0], [0, −1]] (a reflection).
3. Draw each image and find its area by counting squares or with the shoelace formula.
4. Compare each area with |det| and note which maps flip orientation (det < 0). Then check det(AR) = det A det R.

Data table

Matrix	det	Image area	Orientation
A = [[2, 1], [0, 1]]	2	2	kept
R (90° rotation)	1	1	kept
F (reflection)	−1	1	flipped
AR	2	2	kept

Exact values for the unit square.

Sample calculation

1. det A = 2 × 1 − 1 × 0 = 2; the image is a parallelogram with base 2 and height 1.
2. det(AR) = det A × det R = 2 × 1 = 2.

> **Safety**
>
> - None beyond normal desk work.

Home or virtual alternative

- Graph paper and a pencil are all you need.

Interactive Linear Algebra, 3.1 Matrix transformations (https://textbooks.math.gatech.edu/ila/matrix-transformations.html) · Interactive Linear Algebra, 4.3 Determinants and volumes (https://textbooks.math.gatech.edu/ila/determinants-volumes.html)

Planet link: Map projections and image registration use these same matrices; the determinant keeps area measurements of forests and floods honest. USGS: Landsat 9 (https://www.usgs.gov/landsat-missions/landsat-9)

- [ ] **I completed this lab (wet or virtual)**

Lab D: Least-squares fit of sea-level data

WEEK 9

Goal: Fit a line, and then a parabola, to sea-level data with the normal equations, and compare with the spreadsheet's built-in fit.

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{b}, \quad A = [\,\mathbf{1}\ \ \mathbf{t}\,]$$

```
A^TA=A^T, A=[ 1\ \ ]
```

Normal equations for y = C + Dt

Procedure

1. Enter the sample data below (or download NASA's record from its Vital Signs page): t in years since 1993 and h in mm.
2. Build A with a column of ones and a column of t. Compute A^TA and A^Tb with MMULT and TRANSPOSE, then $\hat{x}$ = MINVERSE(A^TA)·A^Tb.
3. Check against LINEST or a chart trendline. Plot the residuals b − A$\hat{x}$.
4. Add a t^2 column and refit. A positive t^2 coefficient means the rise is speeding up.

Data table

t (years since 1993)	0	8	16	24	32
h (mm)	0	22	48	78	114

Sample values shaped like the NASA record (which reports 114.4 mm of rise since 1993); use the real data for a real answer.

Sample calculation

1. $\bar{t} = 16$, $\bar{h} = 52.4$; $\Sigma(t - \bar{t})(h - \bar{h}) = 2272$, $\Sigma(t - \bar{t})^2 = 640$.
2. D = 2272/640 = 3.55 mm/yr; C = 52.4 − 3.55 × 16 = −4.4 mm.
3. The residuals are positive at both ends and negative in the middle: a curve fits better, as NASA's rising rate suggests.

> **Safety**
>
> - Take a short screen break every half hour.

Home or virtual alternative

- The whole lab runs in a free spreadsheet.

NASA: Global mean sea level (https://sealevel.nasa.gov/vital-signs/global-mean-sea-level/) · Interactive Linear Algebra, 6.5 Least squares (https://textbooks.math.gatech.edu/ila/least-squares.html)

Planet link: NASA reports the rate of rise grew from 0.20 cm per year in 1993 to 0.44 cm per year now, an acceleration you can detect with a t^2 column. NASA: Global mean sea level (Vital Signs) (https://sealevel.nasa.gov/vital-signs/global-mean-sea-level/)

- [ ] **I completed this lab (wet or virtual)**

Lab E: A Markov model of materials and recycling

WEEK 12

Goal: Build a transition matrix for material in use and lost to landfill or burning, use matrix powers to follow it over time, and see how a higher recycling rate keeps material in use longer.

$$\mathbf{x}_{k+1} = P\mathbf{x}_k, \quad P = \begin{bmatrix} 1 - f(1-r) & 0 \\ f(1-r) & 1 \end{bmatrix}$$

```
_k+1=P_k, P=1-f(1-r)&0\(1-r)&1
```

Two-state chain: in use, lost

Procedure

1. Choose a sample discard rate f = 0.2 per year and a recovery rate r = 0.321 (EPA's 2018 recycling-plus-composting share).
2. Build P in a spreadsheet and start with x_0 = (1, 0): all material in use.
3. Use MMULT to step forward 10 years, and compare with the formula $(1 - f(1 - r))^k$ for the share still in use.
4. Repeat with r = 0.5 and compare the average years in use, 1/(f(1 − r)).

Data table

Recovery rate r	Stay-in-use probability	In use after 10 years	Average years in use
0.321	0.8642	0.232	7.36
0.50	0.9000	0.349	10.0

Sample model: the discard rate f = 0.2 is a sample value; r = 0.321 is the EPA 2018 share.

Sample calculation

1. Stay probability = 1 − 0.2 × 0.679 = 0.8642; 0.8642^{10} = 0.232.
2. Average years in use = 1/0.1358 = 7.36; at r = 0.5 it is 1/0.1 = 10.0 years.
3. λ = 1 belongs to the "lost" state, which absorbs everything in the long run; λ = 0.8642 sets how fast.

> **Safety**
>
> - Take a short screen break every half hour.

Home or virtual alternative

- The whole lab runs in a free spreadsheet.

Interactive Linear Algebra, 5.6 Stochastic matrices (https://textbooks.math.gatech.edu/ila/stochastic-matrices.html) · EPA: National overview of materials and waste (https://www.epa.gov/facts-and-figures-about-materials-waste-and-recycling/national-overview-facts-and-figures-materials)

Planet link: EPA reports a 32.1% recycling and composting rate in 2018 and 50% landfilled; models like this show what raising that rate is worth. U.S. EPA: National overview, facts and figures on materials, wastes and recycling (https://www.epa.gov/facts-and-figures-about-materials-waste-and-recycling/national-overview-facts-and-figures-materials)

- [ ] **I completed this lab (wet or virtual)**

Lab F: Power iteration and SVD image compression

WEEK 14

Goal: Find a dominant eigenvalue by repeated multiplication, then compress a small grayscale image with a truncated SVD and see how few numbers keep the picture.

$$\mathbf{x}_{k+1} = \frac{A\mathbf{x}_k}{\|A\mathbf{x}_k\|}, \quad A_k = \sum_{i=1}^{k} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$$

```
_k+1=_k\|A_k\|, A_k=_i=1^kσ_i_i_i^T
```

Power iteration and truncated SVD

Procedure

1. In a spreadsheet, start with x_0 = (1, 0) and multiply by A = [[2, 1], [1, 2]] ten times, rescaling each time; watch the ratio of lengths approach 3.
2. Make an 8 × 8 grid of gray levels (0–255) for a simple picture, or load a photo in free Python with NumPy.
3. Compute the SVD (numpy.linalg.svd) and rebuild the image with k = 1, 2 and 4 terms.
4. Record the storage k(8 + 8 + 1) and the share of $\Sigma\sigma^2$ kept, and judge when the picture looks right.

Data table

k	Numbers stored (8 × 8 image)	Share of 64
1	17	26.6%
2	34	53.1%
4	68	106.3%

Exact storage counts. For an 8 × 8 image, k = 4 already stores more than the raw 64 numbers; the savings appear for large images (Unit 10: rank 50 of a 1000 × 1000 image keeps 10.0%).

Sample calculation

1. Power iteration: (1, 0) → (2, 1) → (5, 4) → (14, 13); ‖(14, 13)‖/‖(5, 4)‖ = 19.10/6.403 = 2.98, near λ = 3.
2. Storage for k = 1: 8 + 8 + 1 = 17 numbers, 17/64 = 26.6%.

> **Safety**
>
> - Take a short screen break every half hour.

Home or virtual alternative

- The spreadsheet and free Python versions are both home labs.

MIT OpenCourseWare 18.06 Linear Algebra (Strang, Spring 2010) (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/) · Interactive Linear Algebra, 5.1 Eigenvectors (https://textbooks.math.gatech.edu/ila/eigenvectors.html)

Planet link: NASA JPL's EMIT returns more than 100,000 spectra every second; compressing and separating that data uses exactly these tools. NASA JPL: EMIT, Earth Surface Mineral Dust Source Investigation (https://earth.jpl.nasa.gov/emit/)

☐ **I completed this lab (wet or virtual)**

Exam prep practice sets

Time yourself: about 10 minutes per problem. Then compare with the full solutions.

Midterm practice set (Units 1–5)

1. Find the angle between (1, 0, 1) and (0, 1, 1).

    1. Dot = 1; lengths $\sqrt{2}$ each; $\cos\theta = 1/2$.

    Answer: 60°.

2. Solve $x + 2y = 4$, $3x - y = 5$.

    1. $R_2 - 3R_1$: $-7y = -7$.

    Answer: $x = 2$, $y = 1$.

3. Invert $[[3, 2], [1, 1]]$.

    1. det = 1.

    Answer: $[[1, -2], [-1, 3]]$.

4. A one-sector economy uses 0.4 of its own output. What output meets demand 120?

    1. $x = 120/0.6$.

    Answer: 200.

5. A 6 × 9 matrix has rank 4. Find dim N(A) and dim N(A^T).

    1. $n - r$ and $m - r$.

    Answer: 5 and 2.

Final practice set (all units)

1. Fit y = C + Dt to (0, 0), (1, 2), (2, 2) by least squares.

    1. A^TA = [[3, 3], [3, 5]], A^Tb = (4, 6); det 6.

    Answer: C = 0.333, D = 1.

2. Find det [[2, 1, 0], [1, 2, 1], [0, 1, 2]].

    1. 2(4 − 1) − 1(2 − 0) + 0.

    Answer: 4.

3. Find the eigenvalues of [[6, 2], [2, 3]].

    1. Trace 9, det 14: $\lambda^2 - 9\lambda + 14 = 0$.

    Answer: 7 and 2.

4. Find the steady state of the Markov matrix [[0.9, 0.2], [0.1, 0.8]].

    1. $0.1q_1 = 0.2q_2$.

    Answer: (0.667, 0.333).

5. Find the singular values of [[2, 0], [0, −3]].

    1. A^TA = diag(4, 9).

    Answer: 3 and 2.

6. How many numbers does a rank-20 SVD of a 500 × 800 image store, and what share of the full image is that?

    1. 20(500 + 800 + 1).

    Answer: 26,020, about 6.5%.

Formula sheet

$$\|v\| = \sqrt{v \cdot v},\ \cos\theta = \frac{v \cdot w}{\|v\|\|w\|} \qquad A\mathbf{x} = \mathbf{b},\ \ A = LU$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad (AB)^{-1} = B^{-1}A^{-1},\ (AB)^T = B^TA^T$$

$$\mathbf{x} = (I - A)^{-1}\mathbf{d} \qquad r + \dim N(A) = n$$

$$\mathbf{p} = \frac{a \cdot b}{a \cdot a} a, \;\; A^T A \hat{x} = A^T b$$

$$A = QR, \; Q^T Q = I$$

$$\det(AB) = \det A \det B, \; x_i = \det B_i / \det A$$

$$\det(A - \lambda I) = 0, \; \sum \lambda = \operatorname{tr} A, \; \prod \lambda = \det A$$

$$A = PDP^{-1}, \; A^k = PD^k P^{-1}, \; A = Q \Lambda Q^T$$

$$A = U \Sigma V^T, \; \sigma_i = \sqrt{\lambda_i (A^T A)}, \; \kappa = \sigma_1 / \sigma_n$$

Constants

Constant	Value
Speed of light (used for GPS ranges)	299,792,458 m/s
1 mile	1.609344 km
1 foot	0.3048 m
1 inch	25.4 mm
Flops to solve an n × n system by elimination	about $n^3/3$ multiply–adds (Strang)
Identity matrix I	Ones on the diagonal, zeros elsewhere: AI = IA = A

Source: NIST Guide to the SI, Appendix B.9: conversion factors (https://www.nist.gov/pml/special-publication-811/nist-guide-si-appendix-b-conversion-factors/nist-guide-si-appendix-b9)

Free texts and lectures

Use these free textbooks, lectures and tools alongside the course.

- MIT OpenCourseWare 18.06 Linear Algebra (Strang, Spring 2010) (https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/): Strang's full video lectures, problem sets and exams.
- MIT OpenCourseWare 18.06SC Linear Algebra (self-paced, Fall 2011) (https://ocw.mit.edu/courses/18-06sc-linear-algebra-fall-2011/): Self-paced version with recitation videos and worked problems.
- Interactive Linear Algebra (Margalit and Rabinoff, Georgia Tech) (https://textbooks.math.gatech.edu/ila/): Free interactive textbook used at Georgia Tech.
- Jim Hefferon, Linear Algebra (free text) (https://hefferon.net/linearalgebra/): Free full textbook with answers.
- Gilbert Strang, Introduction to Linear Algebra (book site) (https://math.mit.edu/~gs/linearalgebra/): Strang's book site with notes and figures.
- PhET: Vector Addition (https://phet.colorado.edu/en/simulations/vector-addition): Interactive vectors and components.

Glossary

Basis
A linearly independent set that spans a space.

Column space
All combinations Ax of the columns of A.

Condition number
σ_1/σ_n, how much errors can grow when solving $Ax = b$.

Determinant
Signed volume scale factor of A; zero exactly when A is singular.

Diagonalization
$A = PDP^{-1}$ with eigenvectors in P.

Dimension
Number of vectors in any basis.

Eigenvalue
λ with $Av = \lambda v$ for some nonzero v.

LU factorization
$A = LU$ from elimination.

Least squares
The $\hat{x}$ minimizing $\|b - Ax\|$.

Leontief inverse
$(I - A)^{-1}$, total output needed per unit of final demand.

Linear independence
Only the zero combination gives the zero vector.

Markov matrix
Nonnegative entries, columns summing to 1.

Null space
All x with $Ax = 0$.

Orthonormal
Mutually orthogonal unit vectors.

Pivot
First nonzero entry of a row in echelon form.

Positive definite
Symmetric with $x^TAx > 0$ for $x \neq 0$.

Projection
Closest point to b in a subspace.

Rank
Number of pivots; dimension of the column space.

Singular value
$\sigma_i = \sqrt{\lambda_i(A^TA)}$.

Span
All linear combinations of a set of vectors.

Subspace
A set containing 0 that is closed under addition and scaling.

Self-assessment rubric

No grade or credit is given. Use this to judge honestly where you are in each unit.

Level	What it looks like
4 · Mastery	Chooses the right tool (elimination, projection, eigenvalues or SVD), computes small cases by hand, checks answers (Ax = b, trace and determinant), and explains the meaning in pictures and in context.
3 · Proficient	Solves the practice problems with occasional arithmetic slips.
2 · Developing	Follows worked examples but mixes up rows and columns or the four subspaces.
1 · Beginning	Manipulates symbols without checking sizes or answers; needs to revisit the unit.

Suggested weighting if you want a score

Part	Weight
Practice problems	25%
Labs	20%
Midterm practice set	25%
Final practice set	30%

Aim for level 3 or better in every unit before moving to the next course.

Related courses

- Mechanics of Materials ENGR 213
- Circuits ENGR 214 + ENGR 214L
- Calculus III MATH 210

- All courses