Last updated: November 1, 2026

🗺️ Four-year roadmap · Year 2, Spring

About this course

This free course has 9 units and 4 labs. Those are the real counts, not a longer syllabus cut short.

A check is one number or code result on a course page that a script recomputes before the page is published. The printed number and the recomputed number have to match.

> This is a free self-study course outline written by Saving Planets. It is not a university course, it carries no academic credit, and no school is involved.
>
> Its scope follows a sophomore course in ordinary differential equations, modeled on the topics and depth of MIT OpenCourseWare 18.03 and the free textbooks by Jiří Lebl (Notes on Diffy Qs) and William Trench, and on the syllabi of courses such as Georgia Tech MATH 2552 and Stanford Math 53. We link to the free materials throughout. None of those authors or organizations endorses this page.
>
> What we add: every unit connects to a real planet problem with a cited public source (lakes, methane, fisheries, carbon, climate, epidemics, wind turbines), every number on the page is re-computed by a checking script before it goes live, and four computer labs solve real models numerically.
>
> Work through it in order. Each unit has goals, key ideas, equations, step-by-step examples, practice with full answers, common mistakes, flashcards and a quiz. Your progress is saved only in this browser.

How this course helps save the planet

Almost every model of the living planet is a differential equation: how fast a lake flushes out a pollutant, how a fish stock recovers, how carbon moves between air and sea, how quickly Earth warms when its energy balance shifts, how an epidemic spreads, and how a wind-turbine tower sways.

Each unit ends with a short, sourced section showing how its ideas help both people and nature. Model parameters that are not from a source are labeled as sample values.

For instructors

You can teach this course, or let a student work through it alone. It is free. Saving Planets does not give a grade or school credit.

- This page is the whole course: 9 units and 4 labs (MATH 220).
- Plan on 15 weeks, about 6 to 9 hours a week, including the lab weeks.
- Follow the 15-week schedule. Read the unit, do the practice on paper, then open the answer. In a lab week, do the lab or the home version.
- A check is one number or code result on the page that a script recomputes before we publish. The number you see and the recomputed number have to match.
- Use Print this page, or download the PDF study guide. The formula sheet is near the end.
- Ask: how does this unit help people and the planet? Each unit has a green box with a sourced fact.
- Copy the teacher guide template and write in your class name, the time you have, and the questions you will ask. Open the teacher guide template

Before you start

Prerequisites

- Calculus I and II: derivatives, integrals, partial fractions, Taylor series.
- Matrices, determinants and eigenvalues of 2 × 2 matrices (taught briefly in Unit 6).
- Calculus III helps for the heat equation but is not required.

How to study

- Check every solution by substituting it back into the equation.
- Do every practice problem on paper before opening the answer.
- Sketch a slope field or phase portrait before solving; know what the answer should look like.
- Ask of every model: what are the units, and does the long-run behavior make sense?
- Solve one problem from each unit numerically too, and compare with the formula.
- Retake the quizzes a few days later. Spaced practice beats one long session.

What you need

- A computer or tablet with a spreadsheet.
- A free graphing calculator such as Desmos.
- Optional: Python with NumPy (free in Google Colab).

Suggested 15-week schedule

About 6 to 9 hours a week: reading, practice problems and, in lab weeks, the lab.

Week	Unit	Topics	Reading
1	First-Order Equations	Slope fields; separable equations	Lebl 1.1–1.3
2	First-Order Equations	Linear and exact equations; existence and uniqueness	Lebl 1.2, 1.4, 1.8
3	Modeling with First-Order Equations	Modeling: mixing, decay, logistic growth, harvesting	Lebl 1.6
4	Second-Order Linear Equations and Oscillations	Second-order linear equations; characteristic equation	Lebl 2.1–2.2
5	Second-Order Linear Equations and Oscillations	Mechanical vibrations and damping	Lebl 2.4
6	Forced Vibration and Resonance	Undetermined coefficients; forced oscillation and resonance; Lab D	Lebl 2.5–2.6
7	Laplace Transforms	Laplace transforms; solving IVPs	Lebl 6.1–6.2
8	Laplace Transforms	Step functions, impulses, convolution; midterm practice	Lebl 6.3–6.4
9	Linear Systems, Eigenvalues and Phase Planes	Systems; matrices and eigenvalues	Lebl 3.1–3.4
10	Linear Systems, Eigenvalues and Phase Planes	Phase portraits and classification	Lebl 3.5
11	Nonlinear Systems and Stability	Nonlinear systems, linearization; predator-prey; Lab B	Lebl 8.1–8.3
12	Nonlinear Systems and Stability	Stability; SIR epidemics; Lab A	Lebl 8.2–8.3
13	Numerical Methods	Numerical methods; energy-balance model; Lab C	Lebl 1.7, Trench 3
14	Series Solutions, Fourier Series and the Heat Equation	Series solutions; Fourier series; heat equation	Lebl 7.1–7.2, 4.2, 4.6
15	Review	Review and final practice set	All units

Your progress

Tick units and labs as you finish them, and take the quizzes. Progress and quiz scores are saved only in this browser, never sent to us.

Units

UNIT 1 · WEEKS 1–2

First-Order Equations

Slope fields, separable equations, linear equations with integrating factors, exact equations, and when a solution exists and is unique.

Learning objectives

By the end of this unit you can:

- Read a slope field and sketch solutions.
- Solve separable and linear first-order equations.
- Recognize and solve exact equations.
- State the existence and uniqueness theorem and spot when it fails.

Key concepts

- A differential equation relates an unknown function to its derivatives; a solution is a function that makes it true.
- Separable: dy/dx = g(x)h(y). Divide and integrate both sides.
- Linear: y' + p(x)y = q(x). Multiply by the integrating factor μ = e^(∫p dx) so the left side becomes (μy)'.
- Exact: M dx + N dy = 0 with M_y = N_x is dF = 0, so F(x, y) = C.
- If f and ∂f/∂y are continuous near (x_0, y_0), then y' = f(x, y), $y(x_0) = y_0$ has exactly one solution nearby. Solutions may still blow up in finite time.

Key equations

$$\frac{dy}{h(y)} = g(x)\,dx$$

`(dy)/(h(y))=g(x) dx`

Separable equation

$$\mu = e^{\int p\,dx}, \quad (\mu y)' = \mu q$$

`μ=e^ p dx,(μ y)'=μ q`

Integrating factor

$$M_y = N_x \Rightarrow F(x,y) = C$$

`M_y=N_x F(x,y)=C`

Exact equation

$$y' = f(x, y),\ y(x_0) = y_0$$

`y'=f(x,y),\ y(x_0)=y_0`

Initial value problem

Worked examples, step by step

Example 1.1 Solve dy/dx = xy, y(0) = 2, and find y(1).

1. dy/y = x dx, so ln|y| = x²/2 + C.
2. y = 2e^(x²/2).
3. y(1) = 2e^0.5.

Answer: ≈ 3.297.

Example 1.2 Solve y' + 2y = 4, y(0) = 0, and find y(1).

1. μ = e^(2t), so (e^(2t)y)' = 4e^(2t).
2. e^(2t)y = 2e^(2t) + C; y(0) = 0 gives C = −2.
3. y = 2(1 − e^(−2t)).

Answer: y(1) ≈ 1.729, approaching the steady state 2.

Example 1.3 Solve 2xy dx + (x² + 2y) dy = 0 with y(0) = 1, and find y when x = 1.

1. M_y = 2x = N_x: exact.
2. F = x²y + y²; F(0, 1) = 1.
3. At x = 1: y + y² = 1, so y = (−1 + √5)/2.

Answer: ≈ 0.618.

Practice problems

Work each on paper first, then check.

1. Solve y' = y², y(0) = 1. Where does the solution exist?

   y = 1/(1 − t); it blows up at t = 1, even though y² is smooth everywhere.

2. Solve t y' + y = t² for t > 0.

(ty)' = t², so y = t²/3 + C/t.

3. Solve y' = −y/τ.

   y = y_0e^(−t/τ).

4. Is (y cos x) dx + (sin x + 2y) dy = 0 exact? Solve.

   M_y = cos x = N_x; F = y sin x + y² = C.

5. Does y' = √y, y(0) = 0 have a unique solution?

   No: y = 0 and y = t²/4 both work; ∂f/∂y = 1/(2√y) is not continuous at y = 0.

Common misconceptions

Myth: Every first-order equation is separable.
Actually: y' = x + y is not; it is linear.

Myth: Forgetting +C until the end is fine.
Actually: Add C when you integrate, before solving for y, or the initial condition comes out wrong.

Myth: A smooth right-hand side means the solution lives forever.
Actually: y' = y² blows up at a finite time.

Flashcards

Separable

dy/dx = g(x)h(y).

Linear first-order

y' + p y = q.

Integrating factor

μ = e^(∫p dx).

Exact

M_y = N_x; F = C.

Initial value problem

An ODE plus y(x_0) = y_0.

Existence and uniqueness

f, f_y continuous ⇒ one local solution.

Self-check quiz

1. The integrating factor for y' + 3y = x is...

e^(3x)

e^x

3x

e^(−3x)

e^(3x). e^(∫3 dx).

2. M dx + N dy = 0 is exact when...

M = N

M_y = N_x

M_x = N_y

MN = 1

M_y = N_x. Mixed partials of F agree.

3. y' = ky has solution...

y_0 + kt

y_0e^(kt)

kt^2

sin kt

y_0e^(kt). Separable.

4. Uniqueness can fail when...

f is linear

∂f/∂y is discontinuous

y_0 = 1

the equation is separable

∂f/∂y is discontinuous. e.g. y' = √y at 0.

Try this quiz again

How this helps save the planet

The simplest first-order equation, y' = −y/τ, describes how a pollutant clears from a well-mixed lake or how a gas leaves the atmosphere. Its one number, the time scale τ, decides whether a mistake fades in years or lasts for centuries (Unit 2).

On Saving Planets: Rivers hub

Sources: U.S. EPA: Physical features of the Great Lakes (retention times) (https://www.epa.gov/greatlakes/physical-features-great-lakes) · Lebl, 1.4 Linear equations and the integrating factor (https://www.jirka.org/diffyqs/html/intfactor_section.html)

Free learning resources

- Lebl, 1.3 Separable equations (https://www.jirka.org/diffyqs/html/separable_section.html): Free textbook section.
- Lebl, 1.4 Integrating factor (https://www.jirka.org/diffyqs/html/intfactor_section.html): Linear equations.
- Lebl, 1.8 Exact equations (https://www.jirka.org/diffyqs/html/exact_section.html): Exact equations.
- Lebl, 1.2 Slope fields (https://www.jirka.org/diffyqs/html/slopefields_section.html): Existence and uniqueness.

Watch the lecture

First Order Autonomous Equations (https://ocw.mit.edu/courses/18-03sc-differential-equations-fall-2011/resources/first-order-autonomous-equations/) (in English)

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☐ **I finished this unit**

Modeling with First-Order Equations

Turning a situation into an equation: mixing and lake pollution, decay and lifetimes, Newton cooling, logistic growth, harvesting, and equilibria with their stability.

Learning objectives

By the end of this unit you can:

- Build rate-in minus rate-out models.
- Interpret time constants and lifetimes.
- Find equilibria of autonomous equations and classify their stability.
- Use the logistic model with harvesting to find maximum sustainable yield.

Key concepts

- Rate of change = rate in − rate out. Write the units of every term.
- A well-mixed lake with retention time τ loses a pollutant at rate C/τ once the inflow is clean.
- For an autonomous equation y' = f(y), equilibria solve f(y) = 0; an equilibrium is stable if f'(y) < 0.
- Logistic growth P' = rP(1 − P/K) levels off at K. With constant harvest H, the largest sustainable harvest is rK/4, at P = K/2.

Key equations

$$\frac{dx}{dt} = (\text{rate in}) - (\text{rate out})$$

`dx/dt=(rate in)-(rate out)`

Balance law

$$C(t) = C_0 e^{-t/\tau}$$

`C(t)=C_0e^-t/τ`

Flushing with retention time τ

$$P' = rP\left(1 - \frac{P}{K}\right) - H$$

`P'=rP(1- PK)-H`

Logistic growth with harvest

$$H_{\max} = \frac{rK}{4}$$

H_=rK/4

Maximum sustainable yield

Worked examples, step by step

Example 2.1 Lake Erie's retention time is 2.6 years and Lake Superior's 191 years (EPA). If a pollutant's inflow stops, how long until each lake's concentration falls to 10%, treating each as well mixed?

1. $C/C_0 = e^{(-t/\tau)} = 0.1$, so $t = \tau \ln 10$.
2. Erie: 2.6 × 2.303 = 5.99 years.
3. Superior: 191 × 2.303 = 440 years.

Answer: About 6 years for Erie but about 440 for Superior. Prevention matters most where flushing is slow.

Example 2.2 EPA says methane lasts about a decade in the atmosphere on average. Treat a methane pulse as decaying with $\tau = 10$ years. What fraction is left after 30 years?

1. Fraction = $e^{(-30/10)} = e^{(-3)}$.

Answer: About 5%. Cutting methane leaks therefore pays off within decades, unlike CO_2, which lingers far longer.

Example 2.3 A fish stock follows $P' = 0.4P(1 - P/10{,}000) - H$ (tonnes, years; sample values). Find the maximum sustainable yield.

1. Growth $rP(1 - P/K)$ is largest at $P = K/2 = 5{,}000$ t.
2. There it equals $rK/4 = 0.4 \times 10{,}000/4$.

Answer: 1,000 tonnes per year. Harvesting more than that drives the stock to collapse.

Practice problems

Work each on paper first, then check.

1. Coffee at 90 °C cools in a 20 °C room with $T' = -0.1(T - 20)$ (minutes). Find $T(10)$.

   $T = 20 + 70e^{(-1)} \approx 45.8$ °C.

2. A 100 L tank of fresh water gets brine of 0.5 g/L at 2 L/min and drains at 2 L/min. Find the salt after 50 minutes.

x' = 1 − x/50, x = 50(1 − e^(−t/50)); x(50) = 50(1 − e^(−1)) ≈ 31.6 g.

3. Find and classify the equilibria of y' = y(1 − y).

   y = 0 (f' = 1, unstable) and y = 1 (f' = −1, stable).

4. In the fishery example, what happens if H = 1,200 t/yr?

   rP(1 − P/K) − H < 0 for every P, so the stock declines to zero.

Common misconceptions

Myth: Retention time is how long it takes to remove all the pollutant.

Actually: After one τ, 37% is still there; exponential decay never quite reaches zero.

Myth: More fishing effort always means more catch.

Actually: Past the maximum sustainable yield, more harvest shrinks the stock and future catch.

Myth: An equilibrium is always where the system ends up.

Actually: Only stable equilibria attract nearby solutions.

Flashcards

Balance law

Rate in − rate out.

Time constant

τ in e^(−t/τ); 63% of the change is done after τ.

Retention time

Lake volume ÷ outflow.

Equilibrium

f(y*) = 0.

Stability test

f'(y*) < 0 stable.

Max sustainable yield

rK/4 at P = K/2.

Self-check quiz

1. With τ = 5 years, the fraction left after 5 years is...

0

e^(−1) ≈ 0.37

0.5

0.1

e^(−1) ≈ 0.37. e^(−t/τ) at t = τ.

2. Maximum sustainable yield of logistic growth is...

rK

rK/4

K/2

r/K

rK/4. Peak of rP(1 − P/K).

3. y* is a stable equilibrium of y' = f(y) if...

f'(y*) > 0

f'(y*) < 0

f(y*) > 0

y* > 0

f'(y*) < 0. Nearby solutions are pushed back.

4. Which lake recovers faster from pollution?

Superior (191 yr)

Erie (2.6 yr)

Same

Neither

Erie (2.6 yr). Shorter retention time.

Try this quiz again

How this helps save the planet

The EPA lists Great Lakes retention times from 2.6 years (Erie) to 191 years (Superior), which is why toxins entering Lake Superior are a multi-generation problem.

Fishery scientists at NOAA assess stocks with population models built on these ideas; the logistic model's maximum sustainable yield is the classic starting point.

On Saving Planets: Oceans hub · Rivers hub

Sources: U.S. EPA: Physical features of the Great Lakes (retention times) (https://www.epa.gov/greatlakes/physical-features-great-lakes) · U.S. EPA: Understanding global warming potentials (https://www.epa.gov/ghgemissions/understanding-global-warming-potentials) · NOAA Fisheries: population assessments (https://www.fisheries.noaa.gov/topic/population-assessments)

Free learning resources

- Lebl, 1.6 Autonomous equations (https://www.jirka.org/diffyqs/html/auteq_section.html): Equilibria, logistic growth, harvesting.
- Lebl, 1.4 Integrating factor (https://www.jirka.org/diffyqs/html/intfactor_section.html): Mixing problems.
- OpenStax Calculus 2, 4.4 The Logistic Equation (https://openstax.org/books/calculus-volume-2/pages/4-4-the-logistic-equation): Logistic model.

Watch the lecture

First Order Autonomous Equations (https://ocw.mit.edu/courses/18-03sc-differential-equations-fall-2011/resources/first-order-autonomous-equations/) (in English)

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☐ I finished this unit

Second-Order Linear Equations and Oscillations

Linear equations with constant coefficients, the characteristic equation, and free vibrations: undamped, underdamped, critically damped and overdamped.

Learning objectives

By the end of this unit you can:

- Solve ay'' + by' + cy = 0 with the characteristic equation.
- Handle real, repeated and complex roots.
- Model mass-spring-damper systems.
- Classify damping and compute natural frequency and critical damping.

Key concepts

- Try y = e^(rt): ar² + br + c = 0. Two real roots give two exponentials; a repeated root gives (c_1 + c_2t)e^(rt); complex roots α ± iβ give e^(αt)(c_1 cos βt + c_2 sin βt).
- Superposition: sums of solutions of a homogeneous linear equation are solutions.
- mx'' + cx' + kx = 0 has natural frequency ω_0 = √(k/m).
- Damping c < 2√(mk) oscillates and decays (underdamped); c = 2√(mk) returns fastest without overshoot (critical).

Key equations

$$ar^2 + br + c = 0$$

`ar^2+br+c=0`

Characteristic equation

$$y = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$$

`y=e^α t(c_1β t+c_2β t)`

Complex roots α ± iβ

$$mx'' + cx' + kx = 0,\ \omega_0 = \sqrt{k/m}$$

`mx''+cx'+kx=0,\ ω_0=/m`

Mass-spring-damper

$$c_{\text{crit}} = 2\sqrt{mk}$$

`c_crit=2`

Critical damping

Worked examples, step by step

Example 3.1 Solve y'' + 4y = 0, y(0) = 1, y'(0) = 0. What is the period?

1. $r^2 + 4 = 0$, $r = \pm 2i$.
2. $y = \cos 2t$.
3. Period $2\pi/2$.

Answer: $y = \cos 2t$, period $\pi \approx 3.14$.

Example 3.2 Solve y'' + 2y' + 5y = 0, y(0) = 1, y'(0) = 0, and find y(1).

1. $r = -1 \pm 2i$.
2. $y = e^{-t}(\cos 2t + \tfrac{1}{2} \sin 2t)$ fits both conditions.
3. $y(1) = e^{-1}(\cos 2 + \tfrac{1}{2} \sin 2)$.

Answer: ≈ 0.0142: the oscillation has nearly died out.

Example 3.3 A 2 kg mass hangs on a spring with k = 50 N/m. Find ω_0, the frequency in hertz, and the critical damping.

1. $\omega_0 = \sqrt{(50/2)} = 5$ rad/s.
2. $f = 5/(2\pi) = 0.796$ Hz.
3. $c_{crit} = 2\sqrt{(2 \times 50)} = 20$ N·s/m.

Answer: 5 rad/s, 0.80 Hz, 20 N·s/m.

Practice problems

Work each on paper first, then check.

1. Solve y'' − 3y' + 2y = 0.

   r = 1, 2: $y = c_1 e^t + c_2 e^{2t}$.

2. Solve y'' + 2y' + y = 0.

   Repeated r = −1: $y = (c_1 + c_2 t)e^{-t}$.

3. Classify x'' + 6x' + 5x = 0.

   r = −1, −5: overdamped (c = 6 > 2√5).

4. Find the period of x'' + 9x = 0.

   ω = 3, period 2π/3 ≈ 2.09.

Common misconceptions

Myth: Complex roots mean complex solutions.

Actually: Real and imaginary parts give real solutions: e^(αt)cos βt and e^(αt)sin βt.

Myth: A repeated root gives only one solution.

Actually: The second is t e^(rt).

Myth: More damping always means faster settling.

Actually: Overdamped systems creep back slower than critically damped ones.

Flashcards

Characteristic equation

$ar^2 + br + c = 0$.

Superposition

Sums of solutions are solutions (homogeneous linear).

Repeated root

$(c_1 + c_2t)e^{(rt)}$.

Natural frequency

$\omega_0 = \sqrt{(k/m)}$.

Underdamped

Oscillates while decaying.

Critical damping

$c = 2\sqrt{(mk)}$.

Self-check quiz

1. Roots 2 ± 3i give solutions...

e^(2t), e^(3t)

e^(2t)cos 3t, e^(2t)sin 3t

cos 2t

te^(2t)

e^(2t)cos 3t, e^(2t)sin 3t. α = 2, β = 3.

2. Natural frequency of mx" + kx = 0:

k/m

√(k/m)

m/k

√(mk)

√(k/m). $r^2 = -k/m$.

3. Critical damping c equals...

√(mk)

2√(mk)

mk

k/m

2√(mk). Discriminant zero.

4. y" + 2y' + y = 0 is...

underdamped

critically damped

overdamped

undamped

critically damped. Repeated root.

Try this quiz again

How this helps save the planet

Every tall structure, from a wind-turbine tower to a bridge, has natural frequencies set by its stiffness and mass, exactly $\omega_0 = \sqrt{(k/m)}$ in the simplest model. Damping decides how fast gust-driven sway dies out, which affects fatigue and how long a structure lasts.

On Saving Planets: Physics I, Unit 8: oscillations

Sources: U.S. DOE: How do wind turbines work? (https://www.energy.gov/eere/wind/how-do-wind-turbines-work) · Lebl, 2.4 Mechanical vibrations (https://www.jirka.org/diffyqs/html/sec_mv.html)

Free learning resources

- Lebl, 2.1 Second-order linear ODEs (https://www.jirka.org/diffyqs/html/solinear_section.html): Theory.
- Lebl, 2.2 Constant coefficient equations (https://www.jirka.org/diffyqs/html/sec_ccsol.html): Characteristic equation.
- Lebl, 2.4 Mechanical vibrations (https://www.jirka.org/diffyqs/html/sec_mv.html): Damping.
- PhET: Masses and Springs (https://phet.colorado.edu/en/simulations/masses-and-springs): Simulation.

Watch the lecture

Constant Coefficient Second Order Homogeneous DE's (https://ocw.mit.edu/courses/18-03sc-differential-equations-fall-2011/resources/constant-coefficient-second-order-homogeneous-des/) (in English)

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☐ I finished this unit

Forced Vibration and Resonance

Nonhomogeneous equations, undetermined coefficients, the amplitude response of a driven oscillator, beats and resonance.

Learning objectives

By the end of this unit you can:

- Find particular solutions by undetermined coefficients.
- Write the general solution as homogeneous plus particular.
- Compute the steady-state amplitude of a driven damped oscillator.
- Explain resonance and how engineers avoid it.

Key concepts

- General solution = homogeneous solution + one particular solution.
- Guess the form of the forcing: for cos ωt try A cos ωt + B sin ωt.
- Driven at frequency ω, a damped oscillator settles into steady motion with amplitude X = $F_0/\sqrt{((k - m\omega^2)^2 + (c\omega)^2)}$.
- Near ω = ω_0 the amplitude peaks; with damping ratio ζ = c/c_crit, the peak dynamic amplification is about 1/(2ζ). With no damping, the response grows without bound.

Key equations

$$y = y_h + y_p$$

`y=y_h+y_p`

General solution

$$X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

`X=_0(k-mω^2)^2+(cω)^2`

Steady-state amplitude

$$\frac{X}{F_0/k} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}},\ r = \frac{\omega}{\omega_0}$$

`(X)/(F_0/k)=1(1-r^2)^2+(2 r)^2,\ r=(ω)/(ω_0)`

Dynamic amplification

$$y'' + \omega^2 y = \cos \omega t \Rightarrow y_p = \frac{t \sin \omega t}{2\omega}$$

`y''+ω^2y=ω t y_p=(tω t)/(2ω)`

Pure resonance

Worked examples, step by step

Example 4.1 Find a particular solution of y'' + y = cos 2t.

1. Try y_p = A cos 2t: −4A + A = 1.
2. A = −1/3.

Answer: y_p = −⅓ cos 2t (out of phase, since the drive is above the natural frequency).

Example 4.2 Model a wind-turbine tower as a single oscillator with natural frequency 0.30 Hz and damping ratio ζ = 0.01 (sample values). The rotor turns at 12 rpm = 0.20 Hz, and three blades pass the tower at 0.60 Hz. Find the dynamic amplification at each, and at resonance.

1. At 0.20 Hz, r = 0.667: 1/√((1 − 0.444)² + (0.0133)²) = 1.80.
2. At 0.60 Hz, r = 2: 1/√(9 + 0.0016) = 0.333.
3. At r = 1: 1/(2ζ) = 50.

Answer: 1.8 and 0.33, but 50 at resonance. That is why designers keep a tower's natural frequency away from the rotor and blade-passing frequencies.

Example 4.3 Solve y'' + y = cos t, y(0) = y'(0) = 0.

1. The forcing matches the natural frequency, so try y_p = t(A cos t + B sin t).
2. Substituting gives A = 0, B = ½.
3. y = ½ t sin t.

Answer: The amplitude grows like t/2 without limit: pure resonance.

Practice problems

Work each on paper first, then check.

1. Find y_p for y'' + 3y' + 2y = 4.

$y_p = 2$.

2. Find y_p for $y'' - y = e^t$.

   e^t solves the homogeneous equation, so $y_p = \tfrac{1}{2}\, t\, e^t$.

3. A 1 kg mass, $k = 100$ N/m, $c = 2$ N·s/m is driven by $10 \cos 10t$ N. Find the steady amplitude.

   $k - m\omega^2 = 0$, so $X = 10/(2 \times 10) = 0.5$ m.

4. For $\zeta = 0.05$, what is the peak amplification?

   About $1/(2 \times 0.05) = 10$.

Common misconceptions

Myth: Resonance needs a large force.

Actually: A small periodic force at the natural frequency builds up a large response.

Myth: Damping removes resonance.

Actually: It caps the peak at about $1/(2\zeta)$, which is still large for lightly damped structures.

Myth: Undetermined coefficients works for any forcing.

Actually: Only for polynomials, exponentials, sines and cosines and their products; otherwise use variation of parameters or Laplace.

Flashcards

Particular solution

Any one solution of the forced equation.

Undetermined coefficients

Guess the forcing's form.

Amplitude response

$X = F_0/\sqrt{((k - m\omega^2)^2 + (c\omega)^2)}$.

Damping ratio

$\zeta = c/(2\sqrt{(mk)})$.

Resonance

Large response near ω_0; peak $\approx 1/(2\zeta)$.

Beats

Two close frequencies make a slow envelope.

Self-check quiz

1. General solution =

y_p only

y_h + y_p

y_h × y_p

y_h − y_p

y_h + y_p. Linearity.

2. Resonance occurs when the driving frequency is near...

zero

the natural frequency

twice it

infinity

the natural frequency. Denominator smallest.

3. With ζ = 0.01 the peak amplification is about...

1

10

50

100

50. 1/(2ζ).

4. Driving well above ω_0 gives amplification...

large

less than 1

exactly 1

infinite

less than 1. $\approx 1/r^2$.

Try this quiz again

How this helps save the planet

Wind turbines turn ever larger and taller to harvest more energy, and their towers and blades are lightly damped. Avoiding resonance between the structure and the rotating rotor is a core part of keeping clean-energy machines safe and long-lived. Lab D explores it with a sample model.

On Saving Planets: Physics I, Unit 3: wind power

Sources: U.S. DOE: How do wind turbines work? (https://www.energy.gov/eere/wind/how-do-wind-turbines-work) · Lebl, 2.6 Forced oscillations and resonance (https://www.jirka.org/diffyqs/html/forcedo_section.html)

Free learning resources

- Lebl, 2.5 Nonhomogeneous equations (https://www.jirka.org/diffyqs/html/sec_nonhom.html): Undetermined coefficients and variation of parameters.
- Lebl, 2.6 Forced oscillations and resonance (https://www.jirka.org/diffyqs/html/forcedo_section.html): Resonance.
- MIT OCW 18.03SC, Unit II (https://ocw.mit.edu/courses/18-03sc-differential-equations-fall-2011/): Second-order equations, lectures and problems.

☐ **I finished this unit**

Laplace Transforms

Transforming differential equations into algebra: the transform and its table, derivatives, step functions for switched inputs, impulses, and convolution.

Learning objectives

By the end of this unit you can:

- Compute Laplace transforms from the definition and a table.
- Solve initial value problems with transforms and partial fractions.
- Model switched inputs with unit step functions and shifts.
- Model sudden impulses with the delta function.

Key concepts

- L{f}(s) = ∫₀^∞ e^(−st) f(t) dt turns derivatives into multiplication by s, with initial values built in.
- Solve for Y(s), split with partial fractions, then invert with the table.
- A switch flipped at time a is u(t − a); the shift rule is L{u(t − a)f(t − a)} = e^(−as)F(s).
- An instant kick is δ(t − a), with L{δ(t − a)} = e^(−as).

Key equations

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\, dt$$

`L\=_0^∞ e^-stf(t) dt`

Definition

$$\mathcal{L}\{y'\} = sY - y(0),\ \mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$$

`L'\=sY-y(0),\ L''\=s^2Y-sy(0)-y'(0)`

Derivatives

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a},\ \mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}$$

`L^at\=1s-a,\ L\ω t\=(ω)/(s^2+ω^2)`

Table entries

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$$

`L(t-a)f(t-a)\=e^-asF(s)`

Worked examples, step by step

Example 5.1 Solve y' − 2y = e^(3t), y(0) = 1.

1. sY − 1 − 2Y = 1/(s − 3).
2. Y = 1/((s − 2)(s − 3)) + 1/(s − 2) = [1/(s − 3) − 1/(s − 2)] + 1/(s − 2).
3. Y = 1/(s − 3).

Answer: y = e^(3t). Check: 3e^(3t) − 2e^(3t) = e^(3t).

Example 5.2 A clean lake starts receiving a steady pollutant inflow at t = 2: y' + y = u(t − 2), y(0) = 0. Find y(3).

1. (s + 1)Y = e^(−2s)/s.
2. 1/(s(s + 1)) = 1/s − 1/(s + 1), so y = (1 − e^(−(t − 2)))u(t − 2).
3. y(3) = 1 − e^(−1).

Answer: ≈ 0.632 of the final level, one time constant after the switch.

Example 5.3 A resting oscillator y'' + 4y = δ(t − 1) gets a kick at t = 1. Find y(1 + π/4).

1. (s² + 4)Y = e^(−s).
2. y = ½ sin(2(t − 1)) u(t − 1).
3. At t = 1 + π/4: ½ sin(π/2).

Answer: 0.5, the first peak.

Practice problems

Work each on paper first, then check.

1. Find L{t}.

   1/s².

2. Invert 3/(s² + 9).

   sin 3t.

3. Solve y' + y = 0, y(0) = 4.

$Y = 4/(s + 1)$, $y = 4e^{-t}$.

4. Solve $y'' + y = 0$, $y(0) = 0$, $y'(0) = 2$.

   $Y = 2/(s^2 + 1)$, $y = 2 \sin t$.

5. Find $L\{u(t - 3)\}$.

   e^{-3s}/s.

Common misconceptions

Myth: $L\{fg\} = L\{f\}L\{g\}$.

Actually: The product of transforms corresponds to convolution, not the product.

Myth: Initial conditions are applied at the end.

Actually: They enter immediately through $L\{y'\} = sY - y(0)$.

Myth: Step functions need piecewise solving.

Actually: Laplace handles switches in one pass.

Flashcards

Laplace transform

$\int_0^\infty e^{-st} f\, dt$.

L{y'}

$sY - y(0)$.

$L\{e^{at}\}$

$1/(s - a)$.

Unit step

$u(t - a)$: off then on.

Delta

$\delta(t - a)$: instant impulse; $L = e^{-as}$.

Convolution

$L\{f * g\} = FG$.

Self-check quiz

1. L{y'} =

sY

sY − y(0)

Y/s

Y − s

sY − y(0). Integration by parts.

2. L{e^(−2t)} =

1/(s − 2)

1/(s + 2)

s + 2

−2/s

1/(s + 2). a = −2.

3. A switch turned on at t = 5 is written...

δ(t − 5)

u(t − 5)

e^(5t)

t − 5

u(t − 5). Unit step.

4. L{δ(t)} =

0

1

s

1/s

1. e^(−0s).

Try this quiz again

How this helps save the planet

Real environmental inputs switch on and off: a factory discharge starts, a dam release stops, a volcanic eruption injects aerosols in a single burst. Step and impulse inputs with Laplace transforms give the response to each in one calculation, the method engineers use for control systems in power grids and treatment plants.

On Saving Planets: Rivers hub

Sources: Lebl, 6.1 The Laplace transform (https://www.jirka.org/diffyqs/html/laplace_section.html) · Lebl, 6.4 Dirac delta and impulse response (https://www.jirka.org/diffyqs/html/diracdelta_section.html)

Free learning resources

- Lebl, 6.1 The Laplace transform (https://www.jirka.org/diffyqs/html/laplace_section.html): Definition and table.
- Lebl, 6.2 Transforms of derivatives and ODEs (https://www.jirka.org/diffyqs/html/transformsofders_section.html): Solving IVPs.
- Lebl, 6.3 Convolution (https://www.jirka.org/diffyqs/html/convolution_section.html): Convolution.
- Lebl, 6.4 Dirac delta (https://www.jirka.org/diffyqs/html/diracdelta_section.html): Impulses.

Watch the lecture

Laplace Transform (https://ocw.mit.edu/courses/18-03sc-differential-equations-fall-2011/resources/laplace-transform/) (in English)

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☐ I finished this unit

Linear Systems, Eigenvalues and Phase Planes

Systems x' = Ax, eigenvalue methods, the phase plane, and the classification of equilibria as nodes, saddles, spirals and centers.

Learning objectives

By the end of this unit you can:

- Write coupled equations and higher-order equations as first-order systems.
- Solve x' = Ax with eigenvalues and eigenvectors.
- Sketch phase portraits.
- Classify the origin with the trace and determinant.

Key concepts

- If Av = λv, then x = e^(λt)v solves x' = Ax; the general solution combines one such term per eigenvalue.
- Real eigenvalues of opposite sign: a saddle. Both negative: a stable node.
- Complex eigenvalues α ± iβ: spirals (stable if α < 0); centers if α = 0.
- For a 2 × 2 matrix, λ² − (tr A)λ + det A = 0.

Key equations

$$\mathbf{x}' = A\mathbf{x}$$

`x'=A x`

Linear system

$$\det(A - \lambda I) = 0$$

`(A- I)=0`

Eigenvalues

$$\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

`x=c_1e^_1t v_1+c_2e^_2t v_2`

General solution (distinct real eigenvalues)

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$$

`^2-()+ A=0`

Worked examples, step by step

Example 6.1 A two-box carbon model: atmosphere A and ocean O exchange carbon with A' = −0.2A + 0.1O, O' = 0.2A − 0.1O (per year; sample rates). Start with 3 units in the air and none in the ocean.

1. Eigenvalues: tr = −0.3, det = 0, so λ = 0 and −0.3.
2. λ = 0 has eigenvector (1, 2): the equilibrium split is 1 : 2. Total carbon (3) is conserved, so A → 1, O → 2.
3. A(t) = 1 + 2e^(−0.3t).

Answer: The air keeps one third of the pulse in this toy model, with time scale 1/0.3 ≈ 3.33 years.

Example 6.2 Classify the origin for A = [[1, 2], [2, 1]].

1. $\lambda^2 - 2\lambda - 3 = 0$, λ = 3, −1.

Answer: Opposite signs: a saddle (unstable).

Example 6.3 Classify the origin for A = [[−1, −2], [2, −1]].

1. tr = −2, det = 5: $\lambda^2 + 2\lambda + 5 = 0$.
2. λ = −1 ± 2i.

Answer: A stable spiral: solutions circle inward, decaying like e^(−t).

Practice problems

Work each on paper first, then check.

1. Write y'' + 3y' + 2y = 0 as a system.

   $x_1 = y$, $x_2 = y'$: $x_1' = x_2$, $x_2' = -2x_1 - 3x_2$.

2. Eigenvalues of [[2, 0], [0, −3]]?

   2 and −3: a saddle.

3. Classify [[0, 1], [−4, 0]].

   λ = ±2i: a center (undamped oscillation).

4. Classify [[−3, 1], [0, −2]].

   λ = −3, −2: a stable node.

Common misconceptions

Myth: Eigenvectors are unique.

Actually: Any nonzero multiple works.

Myth: A zero eigenvalue means no motion.

Actually: It means a line of equilibria or a conserved quantity, as in the carbon box model.

Myth: Complex eigenvalues give complex solutions only.

Actually: Real and imaginary parts give real solutions.

Flashcards

Linear system

x' = Ax.

Eigenvalue method

x = e^(λt)v.

Saddle

Real eigenvalues of opposite sign.

Node

Real eigenvalues of the same sign.

Spiral

Complex eigenvalues, α ≠ 0.

Trace–determinant

$\lambda^2 - \operatorname{tr}\lambda + \det = 0$.

Self-check quiz

1. Eigenvalues 2 and −1 give...

a node

a saddle

a spiral

a center

a saddle. Opposite signs.

2. Eigenvalues $-1 \pm 3i$ give…

unstable spiral

stable spiral

center

saddle

stable spiral. $\alpha < 0$.

3. For 2 × 2 A, the eigenvalues' sum is…

det A

tr A

0

1

tr A. $\lambda_1 + \lambda_2 = \text{tr } A$.

4. A center has eigenvalues…

real, equal

purely imaginary

zero

positive

purely imaginary. $\alpha = 0$.

Try this quiz again

How this helps save the planet

Carbon-cycle models are linear systems of boxes (atmosphere, surface ocean, deep ocean, land) exchanging carbon. Their eigenvalues are the time scales on which a CO_2 pulse is absorbed: some fast, some taking centuries or longer. NASA's carbon-cycle overview describes the fast and slow cycles these boxes represent.

On Saving Planets: Oceans hub

Sources: NASA Earth Observatory: The Carbon Cycle (https://earthobservatory.nasa.gov/features/CarbonCycle) · NOAA Global Monitoring Laboratory: Trends in atmospheric CO_2 (https://gml.noaa.gov/ccgg/trends/)

Free learning resources

- Lebl, 3.1 Introduction to systems (https://www.jirka.org/diffyqs/html/sec_introtosys.html): Systems.
- Lebl, 3.4 Eigenvalue method (https://www.jirka.org/diffyqs/html/eigenmethod_section.html): Eigenvalues.
- Lebl, 3.5 Two-dimensional systems and their vector fields (https://www.jirka.org/diffyqs/html/sec_twodimaut.html): Phase portraits.

☐ **I finished this unit**

Nonlinear Systems and Stability

Equilibria of nonlinear systems, linearization with the Jacobian, predator-prey cycles, and the SIR epidemic model.

Learning objectives

By the end of this unit you can:

- Find equilibria of nonlinear systems.
- Linearize with the Jacobian and classify equilibria.
- Analyze the Lotka–Volterra predator-prey model.
- Use the SIR model: R_0, herd-immunity threshold and final size.

Key concepts

- Near an equilibrium, a nonlinear system behaves like its linearization x' = J x, where J is the Jacobian (unless the linearization is a center).
- Lotka–Volterra: prey x' = ax − bxy, predators y' = −cy + dxy. The equilibrium is (c/d, a/b); populations cycle around it.
- SIR: S' = −βSI, I' = βSI − γI, R' = γI. The basic reproduction number R_0 = β/γ; an outbreak grows only if R_0S > 1.
- The herd-immunity threshold is 1 − 1/R_0; the final uninfected share s∞ solves ln s∞ = R_0(s∞ − 1).

Key equations

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

`J=_x&f_y\_x&g_y`

Jacobian

$$x' = ax - bxy,\ y' = -cy + dxy$$

`x'=ax-bxy,\ y'=-cy+dxy`

Lotka–Volterra

$$S' = -\beta SI,\ I' = \beta SI - \gamma I,\ R' = \gamma I$$

`S'=-β SI,\ I'=β SI-γ I,\ R'=γ I`

SIR model

$$R_0 = \beta/\gamma,\ \ln s_\infty = R_0(s_\infty - 1)$$

R_0=β/γ,\ s_∞=R_0(s_∞-1)

Reproduction number and final size

Worked examples, step by step

Example 7.1 Prey and predators follow x' = x − 0.02xy, y' = −0.5y + 0.005xy (sample values, years). Find the coexistence equilibrium and the period of small cycles.

1. x' = 0 needs y = 1/0.02 = 50; y' = 0 needs x = 0.5/0.005 = 100.
2. The Jacobian there has eigenvalues ±i√(ac) = ±i√0.5.
3. Period 2π/√0.5.

Answer: 100 prey and 50 predators; small cycles last about 8.9 years, with predators lagging prey.

Example 7.2 An epidemic has β = 0.3 per day and γ = 0.1 per day. Find R_0, the herd-immunity threshold, and the final share ever infected.

1. R_0 = 3.
2. Threshold 1 − 1/3 = 0.667.
3. Iterate s = e^(3(s − 1)): s∞ = 0.0595.

Answer: R_0 = 3; immunity in 67% stops growth; with no intervention about 94% are eventually infected, more than the threshold because the epidemic overshoots.

Example 7.3 Classify the equilibrium P = K of logistic growth P' = rP(1 − P/K).

1. f'(P) = r − 2rP/K; f'(K) = −r < 0.

Answer: Stable: small disturbances decay at rate r.

Practice problems

Work each on paper first, then check.

1. Find the equilibria of x' = x(3 − x − y), y' = y(2 − x − y).

   (0, 0), (3, 0), (0, 2) (no interior one: the lines x + y = 3 and x + y = 2 are parallel).

2. Linearize $x' = \sin x$ near $x = 0$ and $x = \pi$.

   $f'(0) = 1$ (unstable), $f'(\pi) = -1$ (stable).

3. If $R_0 = 2$, what is the herd-immunity threshold?

   $1 - 1/2 = 50\%$.

4. In the predator-prey example, what happens to the prey if predators are removed?

   $x' = x$: exponential growth (until other limits apply).

Common misconceptions

Myth: Linearization always predicts the behavior.
Actually: Not when it gives a center; the nonlinear terms decide.

Myth: Herd immunity means the epidemic stops at once.
Actually: Infections decline after the threshold but can still overshoot.

Myth: Predator-prey cycles mean the model is unstable.
Actually: Lotka–Volterra orbits are closed: neutrally stable, neither growing nor shrinking.

Flashcards

Jacobian

Matrix of first partials.

Linearization

$x' \approx J\,x$ near an equilibrium.

Lotka–Volterra

Predator-prey cycles around $(c/d, a/b)$.

SIR

Susceptible, infected, recovered.

R_0

β/γ, new infections per case.

Herd-immunity threshold

$1 - 1/R_0$.

Self-check quiz

1. $R_0 = \beta/\gamma = 4$ gives a threshold of...

25%

50%

75%

100%

75%. 1 − 1/4.

2. Lotka–Volterra equilibrium prey level is...

a/b

c/d

b/a

d/c

c/d. $y' = 0$ needs $x = c/d$.

3. Linearization uses...

the Hessian

the Jacobian

the Laplacian

the trace

the Jacobian. First partial derivatives.

4. An epidemic grows initially if...

$R_0 S > 1$

$R_0 < 1$

$\gamma > \beta$

$I = 0$

$R_0S > 1$. $I' = (\beta S - \gamma)I$.

Try this quiz again

How this helps save the planet

On Isle Royale in Lake Superior, the National Park Service charts wolf and moose numbers back to 1980, a long predator-prey record. Their linked ups and downs, and the Park Service's decision to relocate new wolves to the island, are exactly the kind of system Lotka–Volterra models begin to explain.

The same SIR mathematics that guides public-health response is used for wildlife diseases that threaten ecosystems.

On Saving Planets: Oceans hub

Sources: National Park Service: Wolves of Isle Royale (https://www.nps.gov/isro/learn/nature/wolves.htm) · Wolves and Moose of Isle Royale project (long-term data) (https://www.isleroyalewolf.org/)

Free learning resources

- Lebl, 8.1 Linearization, critical points and equilibria (https://www.jirka.org/diffyqs/html/linearization_section.html): Linearization.
- Lebl, 8.2 Stability and classification (https://www.jirka.org/diffyqs/html/nlinstability_section.html): Stability.
- Lebl, 8.3 Applications of nonlinear systems (https://www.jirka.org/diffyqs/html/nlinapps_section.html): Predator-prey and more.

Watch the lecture

The Relation Between Nonlinear Systems and First-order ODE's (https://ocw.mit.edu/courses/18-03sc-differential-equations-fall-2011/resources/the-relation-between-nonlinear-systems-and-first-order-odes/) (in English)

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☐ **I finished this unit**

Numerical Methods

Euler, improved Euler and Runge–Kutta methods, step size and error, and a climate energy-balance model solved numerically.

Learning objectives

By the end of this unit you can:

- Apply Euler, improved Euler (Heun) and RK4.
- Explain the order of a method and how error scales with step size.
- Solve systems numerically.
- Build and solve a zero-dimensional energy-balance climate model.

Key concepts

- Euler steps along the slope: y_{n+1} = y_n + h f(t_n, y_n). Its global error is proportional to h.
- Improved Euler averages two slopes (error ∝ h²); RK4 averages four (error ∝ h⁴).
- Halving h cuts RK4 error about 16 times, which is why it is the workhorse.
- Energy balance: C dT/dt = absorbed sunlight − εσT⁴. Its equilibrium and time scale come from the equation directly.

Key equations

$$y_{n+1} = y_n + hf(t_n, y_n)$$

`y_n+1=y_n+hf(t_n,y_n)`

Euler’s method

$$y_{n+1} = y_n + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

`y_n+1=y_n+ h6(k_1+2k_2+2k_3+k_4)`

Runge–Kutta 4

$$C\frac{dT}{dt} = S(1-\alpha) - \varepsilon\sigma T^4$$

`CdT/dt=S(1-α)-σ T^4`

Energy-balance model

$$\lambda = 4\varepsilon\sigma T^3,\ \tau = C/\lambda$$

`=4σ T^3,\ τ=C/`

Worked examples, step by step

Example 8.1 Approximate y(1) for y' = y, y(0) = 1 with h = 0.1 using Euler, improved Euler and RK4.

1. Each Euler step multiplies by 1.1: $1.1^{10} = 2.5937$.
2. Improved Euler multiplies by 1.105: 2.7141.
3. RK4 multiplies by 1.1051708: 2.7182797.

Answer: True e = 2.7182818. Errors: 0.12, 0.0042 and 0.000002.

Example 8.2 Earth absorbs 340 × (1 − 0.29) = 241.4 W/m² of sunlight (NASA). If it radiated as a blackbody (ε = 1), what temperature would balance it? What effective ε gives today's average of about 288 K?

1. $\sigma T^4 = 241.4$ gives $T = (241.4/5.67 \times 10^{-8})\text{^}(1/4) = 255$ K.
2. $\varepsilon = 241.4/(\sigma \times 288^4) = 0.619$.

Answer: 255 K (−18 °C) without a greenhouse effect; an effective emissivity of 0.62 represents the greenhouse effect that keeps Earth habitable.

Example 8.3 With ε = 0.619 at 288 K and an ocean mixed layer 50 m deep (C = 1,000 × 4,186 × 50 J/(m²·K); sample depth), find λ, the time scale, and the no-feedback warming for a 4 W/m² forcing (sample value).

1. $\lambda = 4\varepsilon\sigma T^3 = 3.35$ W/(m²·K).
2. $\tau = C/\lambda = 6.24 \times 10^7$ s ≈ 1.98 years (about 2).
3. $\Delta T = 4/3.35$.

Answer: About 1.2 K of warming, approached on a 2-year time scale in this simple model. Real feedbacks (water vapor, ice) make the warming larger, as the IPCC assesses.

Practice problems

Work each on paper first, then check.

1. One Euler step for y' = t + y, y(0) = 1, h = 0.1.

   $y_1 = 1 + 0.1(0 + 1) = 1.1$.

2. If Euler's error is 0.12 at h = 0.1, estimate it at h = 0.05.

   About 0.06 (first order).

3. If RK4's error is 2×10^{-6} at h = 0.1, estimate it at h = 0.05.

   About $2 \times 10^{-6}/16 \approx 1.3 \times 10^{-7}$.

4. What equilibrium temperature does the EBM give with $\varepsilon = 1$ and albedo 0.29?

   About 255 K.

Common misconceptions

Myth: Smaller steps are always better.
Actually: Round-off and cost grow; a higher-order method is usually better than tiny Euler steps.

Myth: A numerical answer is exact if the code runs.
Actually: Always check convergence by halving h.

Myth: The 255 K result means the model is wrong.
Actually: It shows how much the greenhouse effect warms Earth: about 33 K.

Flashcards

Euler's method

y + h f.

Improved Euler

Average of two slopes; order 2.

RK4

Four slopes; order 4.

Step size

h; check convergence by halving.

Energy-balance model

$C\,T' = S(1 - \alpha) - \varepsilon\sigma T^4$.

Climate time scale

$\tau = C/\lambda$.

Self-check quiz

1. Euler's method has global error proportional to...

h

h^2

h^4

1/h

h. First order.

2. RK4 is order…

1

2

4

8

4. Error $\propto h^4$.

3. Earth without a greenhouse effect would average about…

288 K

255 K

273 K

300 K

255 K. $\sigma T^4 = 241$ W/m².

4. Halving h in RK4 cuts error by about…

2

4

16

32

16. 2^4.

Try this quiz again

How this helps save the planet

Every climate model, from this one-line energy balance to the global models assessed by the IPCC, is a system of differential equations solved numerically. NASA's energy budget gives the inputs: 340 W/m² of sunlight arrives on average and 29% is reflected.

On Saving Planets: Intro to Programming, simulation unit

Sources: NASA Earth Observatory: Climate and Earth's Energy Budget (https://earthobservatory.nasa.gov/features/EnergyBalance) · IPCC AR6 Working Group I: The Physical Science Basis (https://www.ipcc.ch/report/ar6/wg1/)

Free learning resources

- Lebl, 1.7 Numerical methods: Euler's method (https://www.jirka.org/diffyqs/html/numer_section.html): Euler and error.
- Trench, Ch. 3: Numerical methods (https://digitalcommons.trinity.edu/mono/8/): Improved Euler and RK4.

☐ **I finished this unit**

Series Solutions, Fourier Series and the Heat Equation

Power-series solutions of ODEs, Fourier series of periodic functions, and separation of variables for the heat equation.

Learning objectives

By the end of this unit you can:

- Find power-series solutions from a recurrence.
- Compute Fourier coefficients of simple periodic functions.
- Solve the heat equation on a rod by separation of variables.
- Estimate how deep seasonal temperature waves reach into the ground.

Key concepts

- Substitute y = Σ$a_n x^n$, match powers, and get a recurrence for the coefficients.
- A periodic function is a sum of sines and cosines; for an odd function on (−π, π), $b_n = (2/\pi)\int_0^{\pi} f \sin nx\, dx$.
- The heat equation $u_t = \alpha u_{xx}$ spreads heat; each sine mode decays like $e^{-\alpha(n\pi/L)^2 t}$, fast modes first.
- A periodic surface temperature penetrates the ground with damping depth $d = \sqrt{2\alpha/\omega}$.

Key equations

$$y = \sum a_n x^n$$

`y= a_nx^n`

Power-series solution

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

`b_n=2π_0^π f(x) nx dx`

Fourier sine coefficient

$$u_t = \alpha u_{xx}$$

`u_t=α u_xx`

Heat equation

$$d = \sqrt{2\alpha/\omega}$$

Worked examples, step by step

Example 9.1 Solve y'' + y = 0 by power series.

1. $\Sigma(n+2)(n+1)a_{n+2}x^n + \Sigma a_n x^n = 0$.
2. $a_{n+2} = -a_n/((n+2)(n+1))$.
3. Even terms build cos x, odd terms sin x.

Answer: $y = a_0 \cos x + a_1 \sin x$, recovered from the recurrence.

Example 9.2 Find the Fourier series of the square wave f = 1 on (0, π), −1 on (−π, 0).

1. Odd, so only sines: $b_n = (2/\pi)\int_0^{\wedge}\pi \sin nx\, dx = (2/(n\pi))(1 - \cos n\pi)$.
2. $b_n = 4/(n\pi)$ for odd n, 0 for even n.

Answer: $f = (4/\pi)(\sin x + \sin 3x/3 + \sin 5x/5 + \ldots)$; $b_1 \approx 1.273$.

Example 9.3 Soil with diffusivity $\alpha = 5 \times 10^{-7}$ m²/s (sample value) feels a yearly surface temperature cycle. Find the damping depth.

1. $\omega = 2\pi/(3.156 \times 10^7 \text{ s}) = 1.99 \times 10^{-7} \text{ s}^{-1}$.
2. $d = \sqrt{(2 \times 5 \times 10^{-7}/1.99 \times 10^{-7})}$.

Answer: About 2.24 m: the yearly swing shrinks by a factor e for each 2.2 m of depth, so deeper ground responds to warming slowly, over many years.

Practice problems

Work each on paper first, then check.

1. Find the recurrence for y' = y by power series.

   $a_{n+1} = a_n/(n+1)$, so $y = a_0 e^x$.

2. Is $f(x) = x^2$ odd or even? Which Fourier terms appear?

   Even: cosines only (plus a constant).

3. A rod mode decays like $e^{\wedge}(-\alpha(\pi/L)^2 t)$. How does doubling L change its decay time?

It multiplies the decay time by 4.

4. Find b_3 for the square wave.

   $4/(3\pi) \approx 0.424$.

Common misconceptions

Myth: Fourier series only work for smooth functions.

Actually: They work for jumps too, converging to the midpoint at a jump.

Myth: Heat diffuses at a fixed speed.

Actually: Diffusion time grows with distance squared.

Myth: Series solutions are always better than formulas.

Actually: They are most useful when no closed form exists, as for Airy or Bessel equations.

Flashcards

Power-series method

Substitute $\Sigma a_n x^n$; get a recurrence.

Fourier series

Sum of sines and cosines.

Fourier coefficient

$b_n = (2/\pi)\int_0^\pi f \sin nx\, dx$.

Heat equation

$u_t = \alpha\, u_{xx}$.

Separation of variables

$u = X(x)T(t)$.

Damping depth

$\sqrt{(2\alpha/\omega)}$.

Self-check quiz

1. In the heat equation, higher modes decay...

slower

faster

not at all

linearly

faster. Rate ∝ n^2.

2. An odd function's Fourier series has only...

cosines

sines

constants

exponentials

sines. Odd × even integrals vanish.

3. Doubling the period of a surface temperature cycle changes the damping depth by...

2

√2

1/2

4

√2. d ∝ 1/√ω.

4. b_1 for the square wave is...

1

4/π

2/π

π/4

4/π. 4/(1·π).

Try this quiz again

How this helps save the planet

Permafrost in the Northern Hemisphere holds about 1,672 billion tons of organic carbon, according to NASA. How fast surface warming reaches it is a heat-equation problem, and the damping depth shows why deeper layers respond slowly but keep responding long after the surface warms.

On Saving Planets: Calculus III

Sources: NASA Earth Observatory: The Carbon Cycle (https://earthobservatory.nasa.gov/features/CarbonCycle) · Lebl, 4.6 PDEs, separation of variables, and the heat equation (https://www.jirka.org/diffyqs/html/heateq_section.html)

Free learning resources

- Lebl, 7.1 Power series (https://www.jirka.org/diffyqs/html/powerseries_section.html): Power series.
- Lebl, 7.2 Series solutions of linear second-order ODEs (https://www.jirka.org/diffyqs/html/seriessols_section.html): Series solutions.
- Lebl, 4.2 Trigonometric series (https://www.jirka.org/diffyqs/html/ts_section.html): Fourier series.
- Lebl, 4.6 The heat equation (https://www.jirka.org/diffyqs/html/heateq_section.html): Heat equation.

Watch the lecture

Convergence of Fourier Series (https://ocw.mit.edu/courses/18-03sc-differential-equations-fall-2011/resources/convergence-of-fourier-series/) (in English)

MIT 18.03SC Differential Equations (Fall 2011), MIT OpenCourseWare. License: CC BY-NC-SA 4.0 (https://creativecommons.org/licenses/by-nc-sa/4.0/) · Terms of use (https://ocw.mit.edu/pages/privacy-and-terms-of-use/). Linked for free study only; Saving Planets is not affiliated with or endorsed by this university.

☐ **I finished this unit**

Problem Sets

These practice problems live inside each unit. Work on paper first, then open the answer.

1. First-Order Equations
   5 problems
2. Modeling with First-Order Equations
   4 problems
3. Second-Order Linear Equations and Oscillations
   4 problems
4. Forced Vibration and Resonance
   4 problems
5. Laplace Transforms
   5 problems
6. Linear Systems, Eigenvalues and Phase Planes
   4 problems

7. Nonlinear Systems and Stability
   4 problems
8. Numerical Methods
   4 problems
9. Series Solutions, Fourier Series and the Heat Equation
   4 problems

Computer labs

Four computer labs: an SIR epidemic solved with Euler and RK4, a predator-prey phase plane, a climate energy-balance model, and a frequency sweep of a wind-turbine tower model.

Sample parameters are labeled as such; each lab links to a public source for the real system.

Safety reference: Lebl: Notes on Diffy Qs (free textbook) (https://www.jirka.org/diffyqs/)

Lab A: An SIR epidemic with Euler and RK4

WEEK 12

Goal: Solve the SIR model numerically, find the peak, and compare Euler with RK4 against the exact final size.

$$S' = -0.3SI,\ I' = 0.3SI - 0.1I,\ R' = 0.1I$$

`S'=-0.3SI,\ I'=0.3SI-0.1I,\ R'=0.1I`

SIR with $R_0 = 3$ (sample rates per day)

Procedure

1. Start with S = 0.999, I = 0.001, R = 0 (fractions of the population).
2. Run Euler with h = 1 day and h = 0.1 day to day 160, then RK4 with h = 1 day.
3. Record the final R and the peak of I and when it occurs; compare with theory.

Data table

Method	Final R (day 160)
Euler, h = 1	0.9467
Euler, h = 0.1	0.9411
RK4, h = 1	0.9405
Exact final size (theory)	0.9405

Computed values for sample rates; not a real outbreak.

Sample calculation

1. The peak is 30.1% infected at once, near day 38.4; theory gives I_max = 1 − (1 + ln R_0)/R_0 = 0.300.
2. Euler with h = 1 overstates the final size by 0.006; RK4 with the same step matches theory to 4 digits.

> **Safety**
>
> - None beyond normal computer use.

Home or virtual alternative

- A spreadsheet works; Python is easier for RK4.

Lebl, 8.3 Applications of nonlinear systems (https://www.jirka.org/diffyqs/html/nlinapps_section.html)

Planet link: The same model structure is used for disease outbreaks in people and in wildlife.

- [ ] **I completed this lab (wet or virtual)**

Lab B: The predator-prey phase plane

WEEK 11

Goal: Integrate the Lotka–Volterra model with RK4 and plot the closed orbit in the phase plane.

$$x' = x - 0.02xy, \quad y' = -0.5y + 0.005xy$$

```
x'=x-0.02xy, y'=-0.5y+0.005xy
```

Lotka–Volterra (sample values)

Procedure

1. Start at 150 prey and 50 predators.
2. Use RK4 with a small step (0.001 years in Python, or 0.01 in a spreadsheet) for 40 years.
3. Plot y against x and record the prey range and the period.

Data table

Quantity	Value
Equilibrium	(100, 50)
Prey range	62.6 to 150
Period	8.99 years
Small-cycle period (linear theory)	8.89 years

Computed values; sample parameters.

Sample calculation

1. The orbit closes: the populations repeat every 8.99 years, slightly longer than the 8.89 years linear theory predicts for small cycles, because this cycle is large.
2. Predators peak after prey, a lag seen in many real predator-prey records.

> **Safety**
>
> - None beyond normal computer use.

Home or virtual alternative

- Desmos can draw the phase plane slope field.

Wolves and Moose of Isle Royale project (long-term data) (https://www.isleroyalewolf.org/)

Planet link: Long-term records such as Isle Royale's wolves and moose test these models. National Park Service: Wolves of Isle Royale (https://www.nps.gov/isro/learn/nature/wolves.htm)

- [ ] **I completed this lab (wet or virtual)**

Lab C: A climate energy-balance model

WEEK 13

Goal: Step a one-box climate model through time after a sudden forcing and find its time scale.

$$C\frac{d\Delta T}{dt} = F - \lambda\,\Delta T$$

```
C(dΔ T)/(dt)=F- Δ T
```

Linearized energy-balance model

Procedure

1. Use C = 2.09 × 10⁸ J/(m²·K) (50 m of ocean), λ = 3.35 W/(m²·K) from Unit 8, and a sudden forcing F = 4 W/m² (sample value).
2. Step with Euler using monthly steps for 10 years.
3. Compare with the exact solution ΔT = (F/λ)(1 − e^(−t/τ)).

Data table

Year	ΔT (K)
1	0.48
2	0.77
5	1.10
10	1.19

Computed model values; no feedbacks other than radiation.

Sample calculation

1. Equilibrium ΔT = F/λ = 1.19 K; τ = C/λ ≈ 2.0 years, so the warming is about 63% done after 2 years in this model.
2. Real climate responds more slowly and warms more, because heat mixes into the deep ocean and feedbacks amplify it.

> **Safety**
>
> - None beyond normal computer use.

Home or virtual alternative

- A spreadsheet is enough.

NASA Earth Observatory: Climate and Earth's Energy Budget (https://earthobservatory.nasa.gov/features/EnergyBalance)

Planet link: Energy-balance models are the first step toward the climate projections assessed by the IPCC. IPCC AR6 Working Group I: The Physical Science Basis (https://www.ipcc.ch/report/ar6/wg1/)

- [ ] **I completed this lab (wet or virtual)**

Lab D: Resonance sweep of a wind-turbine tower model

WEEK 6

Goal: Compute the dynamic amplification of a single-oscillator tower model across driving frequencies.

$$A(f) = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}},\ r = f / f_0$$

`A(f)=1(1-r^2)^2+(2 r)^2,\ r=f/f_0`

Dynamic amplification

Procedure

1. Use f_0 = 0.30 Hz and ζ = 0.01 (sample values).
2. Tabulate A for driving frequencies from 0.1 to 0.6 Hz.
3. Mark the rotor frequency (0.20 Hz at 12 rpm) and the blade-passing frequency (0.60 Hz) and judge the design.

Data table

Driving frequency (Hz)	Amplification
0.10	1.12
0.20	1.80
0.25	3.27
0.30	50.0
0.35	2.76
0.60	0.333

Computed values for a sample model, not a specific turbine.

Sample calculation

1. Both operating frequencies sit well away from the 0.30 Hz peak, where the response would be 50 times the static deflection.
2. If the rotor sped up to 18 rpm (0.30 Hz), it would drive the tower at resonance; this is why designers check the natural frequency against the whole speed range.

> **Safety**

- None beyond normal computer use.

Home or virtual alternative

- A spreadsheet is enough.

PhET: Masses and Springs (https://phet.colorado.edu/en/simulations/masses-and-springs)

Planet link: Avoiding resonance keeps wind turbines generating clean power safely for decades.
U.S. DOE: How do wind turbines work? (https://www.energy.gov/eere/wind/how-do-wind-turbines-work)

☐ **I completed this lab (wet or virtual)**

Exam prep practice sets

Time yourself: about 10 minutes per problem. Then compare with the full solutions.

Midterm practice set (Units 1–5)

1. Solve y' = −2ty, y(0) = 3, and find y(1).

    1. dy/y = −2t dt; y = 3e^(−t²).

    Answer: 3e^(−1) ≈ 1.104.

2. A lake with retention time 4 years: how long until a pollutant falls to half after inflow stops?

    1. t = 4 ln 2.

    Answer: ≈ 2.77 years.

3. Solve y'' + 6y' + 9y = 0, y(0) = 1, y'(0) = 0, and find y(1).

    1. y = (1 + 3t)e^(−3t).

    Answer: 4e^(−3) ≈ 0.199.

4. Solve y' + y = 1, y(0) = 0 by Laplace and find y(2).

    1. Y = 1/(s(s + 1)) = 1/s − 1/(s + 1).

    Answer: 1 − e^(−2) ≈ 0.865.

Final practice set (all units)

1. Classify the origin of x' = [[0, 1], [−2, −2]]x.

    1. λ² + 2λ + 2 = 0, λ = −1 ± i.

Answer: Stable spiral.

2. An SIR epidemic has β = 0.5, γ = 0.25 per day. Find R_0 and the herd-immunity threshold.

    1. $R_0 = 2$.

    Answer: $R_0 = 2$, threshold 50%.

3. Two Euler steps with h = 0.5 for y' = −y, y(0) = 1.

    1. Each step multiplies by 0.5.

    Answer: y(1) ≈ 0.25 (true e^(−1) ≈ 0.368).

4. A mass-spring system with m = 1 kg, k = 16 N/m, c = 0.4 N·s/m. Find ω_0 and ζ.

    1. $\omega_0 = 4$ rad/s; c_crit = 8.

    Answer: $\omega_0 = 4$ rad/s, ζ = 0.05 (peak amplification about 10).

Formula sheet

$$\mu = e^{\int p\,dx}$$

$$M_y = N_x \Rightarrow F = C$$

$$C = C_0 e^{-t/\tau}$$

$$H_{\max} = rK/4$$

$$ar^2 + br + c = 0$$

$$\omega_0 = \sqrt{k/m},\ c_{\text{crit}} = 2\sqrt{mk}$$

$$X = F_0 / \sqrt{(k - m\omega^2)^2 + (c\omega)^2}$$

$$\mathcal{L}\{y'\} = sY - y(0)$$

$$\mathcal{L}\{e^{at}\} = 1/(s-a)$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$$

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$$

$$R_0 = \beta/\gamma,\ 1 - 1/R_0$$

$$y_{n+1} = y_n + hf$$

$$C\,T' = S(1-\alpha) - \varepsilon\sigma T^4$$

$$u_t = \alpha u_{xx}$$

$$b_n = \tfrac{2}{\pi}\int_0^{\pi} f \sin nx\,dx$$

Constants

Constant	Value
Stefan–Boltzmann constant σ	5.67×10^{-8} W/(m²·K⁴)
Average sunlight at the top of the atmosphere	340 W/m²
Fraction of sunlight reflected	29%
Lake Superior retention time	191 years
Lake Erie retention time	2.6 years
Methane's average lifetime	about a decade

Source: NASA Earth Observatory: Climate and Earth's Energy Budget (https://earthobservatory.nasa.gov/features/EnergyBalance)

Free texts and tools

Use these free textbooks and tools alongside the course.

- Lebl: Notes on Diffy Qs (https://www.jirka.org/diffyqs/): Free, complete textbook for this course.
- Trench: Elementary Differential Equations (free text) (https://digitalcommons.trinity.edu/mono/8/): Free textbook with many exercises.
- MIT OpenCourseWare 18.03SC Differential Equations (https://ocw.mit.edu/courses/18-03sc-differential-equations-fall-2011/): Free MIT lectures, recitations and problem sets.
- Desmos graphing calculator (https://www.desmos.com/calculator): Graph solutions and slope fields.
- PhET: Masses and Springs (https://phet.colorado.edu/en/simulations/masses-and-springs): Damping and oscillation intuition.

Glossary

Autonomous equation
y' = f(y), with no explicit t.

Characteristic equation
$ar^2 + br + c = 0$ for constant-coefficient ODEs.

Critical damping
Fastest return without overshoot.

Eigenvalue
λ with Av = λv.

Energy-balance model
Climate model balancing absorbed sunlight and emitted heat.

Equilibrium
Constant solution.

Euler’s method
First-order numerical stepping.

Exact equation
M dx + N dy = 0 with M_y = N_x.

Fourier series
Expansion in sines and cosines.

Heat equation
u_t = α u_xx.

Integrating factor
Multiplier making a linear ODE exact.

Jacobian
Matrix of first partials used to linearize.

Laplace transform
∫₀^∞ e^(−st) f dt.

Lotka–Volterra
Predator-prey model.

Phase portrait
Trajectories in the state plane.

Resonance
Large response when forcing matches a natural frequency.

Runge–Kutta (RK4)
Fourth-order numerical method.

R_0
Basic reproduction number.

SIR model
Susceptible–infected–recovered epidemic model.

Separable equation
dy/dx = g(x)h(y).

Self-assessment rubric

No grade or credit is given. Use this to judge honestly where you are in each unit.

Level	What it looks like
4 · Mastery	Builds models from a description, solves them by formula and numerically, checks units and long-run behavior.
3 · Proficient	Solves the practice problems with occasional slips.

Level	What it looks like
2 · Developing	Follows worked examples but struggles to set up models or choose a method.
1 · Beginning	Applies formulas without checking solutions; needs to revisit the unit.

Suggested weighting if you want a score

Part	Weight
Practice problems	30%
Computer labs	15%
Midterm practice set	25%
Final practice set	30%

Aim for level 3 or better in every unit before moving to the next course.

Related courses

- Calculus III MATH 210
- Physics II PHYS 210 + PHYS 210L
- Statics ENGR 211
- All courses