Last updated: November 1, 2026

🗺️ Four-year roadmap · Year 1, Fall

About this course

This free course has 6 units and 4 labs. Those are the real counts, not a longer syllabus cut short.

A check is one number or code result on a course page that a script recomputes before the page is published. The printed number and the recomputed number have to match.

> This is a free self-study course outline written by Saving Planets. It is not a university course, it carries no academic credit, and no school is involved.
>
> Its scope follows first-semester college calculus (single-variable differential calculus and an introduction to integrals), modeled on the topics and depth of open materials such as OpenStax Calculus Volume 1 and MIT OpenCourseWare 18.01SC Single Variable Calculus, which we link to throughout. Those organizations do not endorse this page.
>
> What we add: every unit connects calculus to real climate, energy and spaceflight problems, every number on the page is re-computed by a checking script before it goes live, and four computer labs let you do calculus on data the way scientists and engineers do.
>
> Work through it in order. Each unit has goals, key ideas, equations, step-by-step examples, practice with full answers, common mistakes, flashcards and a quiz. Your progress is saved only in this browser.

How this course helps save the planet

Calculus is the mathematics of change. Climate scientists use derivatives to turn a record of carbon dioxide or sea ice into a rate of change, engineers use optimization to get the most energy from wind and the most mission from every kilogram of rocket fuel, and integrals add up power into energy.

Each unit ends with a short, sourced section showing how its ideas help both people and nature.

For instructors

You can teach this course, or let a student work through it alone. It is free. Saving Planets does not give a grade or school credit.

- This page is the whole course: 6 units and 4 labs (MATH 110).
- Plan on 15 weeks, about 6 to 9 hours a week, including the lab weeks.
- Follow the 15-week schedule. Read the unit, do the practice on paper, then open the answer. In a lab week, do the lab or the home version.
- A check is one number or code result on the page that a script recomputes before we publish. The number you see and the recomputed number have to match.
- Use Print this page, or download the PDF study guide. The formula sheet is near the end.
- Ask: how does this unit help people and the planet? Each unit has a green box with a sourced fact.
- Copy the teacher guide template and write in your class name, the time you have, and the questions you will ask. Open the teacher guide template

Before you start

Prerequisites

- Algebra: factoring, rational expressions, solving equations and inequalities.
- Functions and graphs: domain, range, composition, inverse functions.
- Trigonometry: radians, the unit circle, sine, cosine, tangent and basic identities.
- Exponential and logarithmic functions, including e and ln.

How to study

- Read the unit, then close it and try the worked examples yourself before checking each step.
- Do every practice problem on paper before opening the answer.
- Sketch a graph whenever you can. Most calculus mistakes show up as a picture that makes no sense.
- Check derivatives by estimating the slope numerically, and integrals by estimating the area.
- Use a graphing tool such as Desmos to test your intuition, never to replace the work.
- Retake the quizzes a few days later. Spaced practice beats one long session.

What you need

- A computer or tablet with a spreadsheet (Google Sheets, Excel or LibreOffice).
- A free graphing calculator such as Desmos.
- Optional: Python with NumPy, for those who want to code the labs.

Suggested 15-week schedule

About 6 to 9 hours a week: reading, practice problems and, in lab weeks, the lab.

Week	Unit	Topics	Reading
1	Functions & Limits	Functions review; the idea of a limit; limits from graphs and tables	OpenStax 1.1–1.5, 2.1–2.2
2	Functions & Limits	Limit laws; algebraic techniques; special trigonometric limits	OpenStax 2.3
3	Functions & Limits	Limits at infinity; asymptotes; the precise definition	OpenStax 2.5, 4.6
4	Continuity & the Derivative	Continuity, the IVT, the derivative; Lab A	OpenStax 2.4, 3.1–3.2
5	Differentiation Rules	Power, product and quotient rules; higher derivatives	OpenStax 3.3–3.4
6	Differentiation Rules	Trigonometric derivatives; the chain rule	OpenStax 3.5–3.6
7	Differentiation Rules	Implicit differentiation; exponential and log derivatives; midterm practice	OpenStax 3.7–3.9
8	Applications I: Related Rates, Approximation & Newton's Method	Related rates; linear approximation and differentials	OpenStax 4.1–4.2
9	Applications I: Related Rates, Approximation & Newton's Method	L'Hôpital's rule; Newton's method; Lab B	OpenStax 4.8–4.9
10	Applications II: Extrema, Graph Shape & Optimization	Extrema, the Mean Value Theorem, graph shape	OpenStax 4.3–4.5
11	Applications II: Extrema, Graph Shape & Optimization	Applied optimization; Lab C	OpenStax 4.7
12	Integrals & the Fundamental Theorem of Calculus	Riemann sums and the definite integral	OpenStax 5.1–5.2
13	Integrals & the Fundamental Theorem of Calculus	The Fundamental Theorem; Lab D	OpenStax 5.3–5.4
14	Integrals & the Fundamental Theorem of Calculus	Substitution; average value; areas between curves	OpenStax 5.5, 6.1
15	Review	Review and final practice set	All units

Your progress

Tick units and labs as you finish them, and take the quizzes. Progress and quiz scores are saved only in this browser, never sent to us.

Units

Functions & Limits

A limit describes what a function approaches. It is the idea underneath every derivative and integral.

Learning objectives

By the end of this unit you can:

- Estimate limits from graphs and tables, including one-sided limits.
- Evaluate limits with the limit laws, factoring and rationalizing.
- Find limits at infinity and identify horizontal and vertical asymptotes.
- State the precise (ε–δ) definition of a limit.

Key concepts

- lim f(x) as x → a = L means f(x) can be made as close to L as we like by taking x close enough to a, but not equal to a.
- A limit exists only if the left- and right-hand limits agree.
- For a 0/0 form, simplify first: factor, cancel, or multiply by a conjugate.
- Two special limits: sin x / x → 1 and (1 − cos x)/x → 0 as x → 0 (x in radians).
- At infinity, compare the highest powers in a rational function.

Key equations

$$\lim_{x \to a} f(x) = L$$

```
_x af(x)=L
```

Limit notation

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

```
_x0( x)/(x)=1
```

A fundamental trigonometric limit

$$\forall \varepsilon > 0 \; \exists \delta > 0 : \; 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Precise definition of a limit

Worked examples, step by step

Example 1.1 Find $\lim (x^2 - 4)/(x - 2)$ as $x \to 2$.

1. Direct substitution gives 0/0, so simplify.
2. $(x^2 - 4)/(x - 2) = (x - 2)(x + 2)/(x - 2) = x + 2$ for $x \neq 2$.
3. Now substitute: $2 + 2 = 4$.

Answer: 4.

Example 1.2 Find $\lim (\sqrt{(x + 4)} - 2)/x$ as $x \to 0$.

1. 0/0 again. Multiply top and bottom by the conjugate $\sqrt{(x + 4)} + 2$.
2. Top becomes $(x + 4) - 4 = x$, so the expression is $1/(\sqrt{(x + 4)} + 2)$.
3. Substitute $x = 0$: $1/(2 + 2) = 1/4$.

Answer: 1/4.

Example 1.3 Find $\lim (3x^2 + 1)/(2x^2 - 5)$ as $x \to \infty$.

1. Divide top and bottom by x^2: $(3 + 1/x^2)/(2 - 5/x^2)$.
2. As $x \to \infty$, $1/x^2 \to 0$.

Answer: 3/2, so $y = 3/2$ is a horizontal asymptote.

Practice problems

Work each on paper first, then check.

1. $\lim (x^2 - 9)/(x - 3)$ as $x \to 3$

   Factor: $x + 3 \to 6$.

2. $\lim (5x + 2)/(x - 1)$ as $x \to \infty$

   Divide by x: 5.

3. $\lim (1 - \cos x)/x$ as $x \to 0$

   0 (a standard limit).

4. $\lim 1/x$ as $x \to 0^+$ and as $x \to 0^-$

   $+\infty$ from the right, $-\infty$ from the left, so the two-sided limit does not exist.

5. $\lim \sin(3x)/x$ as $x \to 0$

   Write it as $3 \cdot \sin(3x)/(3x) \to 3 \times 1 = 3$.

Common misconceptions

Myth: A limit is just the value of the function at that point.

Actually: The limit ignores f(a). f can be undefined at a and still have a limit there, as in Example 1.1.

Myth: 0/0 means the limit is 0 (or does not exist).

Actually: 0/0 is an indeterminate form: the limit could be anything. You have to simplify.

Myth: If the graph gets close to a line, it can never cross it.

Actually: A graph can cross a horizontal asymptote; the asymptote describes behavior far away.

Flashcards

Limit

The value f(x) approaches as x approaches a.

One-sided limit

Limit from the left ($x \to a^-$) or right ($x \to a^+$).

Indeterminate form

0/0 or ∞/∞: simplify before deciding.

Horizontal asymptote

$y = L$ when $f(x) \to L$ as $x \to \pm\infty$.

Vertical asymptote

$x = a$ when $f(x) \to \pm\infty$ as $x \to a$.

ε–δ definition

For every $\varepsilon > 0$ there is $\delta > 0$ that keeps f(x) within ε of L.

Self-check quiz

1. $\lim (x^2 - 1)/(x - 1)$ as $x \to 1$ is...

0

1

2

undefined

2. Factor to x + 1 → 2.

2. A two-sided limit exists when...

f(a) is defined

the left and right limits are equal

the function is a polynomial

the graph has no holes

the left and right limits are equal. Both one-sided limits must exist and agree.

3. lim sin x / x as x → 0 (radians) equals...

0

1

∞

π/180

1. A fundamental limit; it is why calculus uses radians.

4. lim $(4x^3 - x)/(2x^3 + 7)$ as x → ∞ is...

0

2

4

∞

2. Leading coefficients: 4/2 = 2.

Try this quiz again

How this helps save the planet

Every climate record is a list of measurements at separate times. The idea of a limit, shrinking the time step toward zero, is what lets scientists talk about the rate of change at an instant rather than just over a year or a decade.

NOAA has measured carbon dioxide at Mauna Loa Observatory since 1958. Its data pages publish both the concentration and its yearly growth rate, a direct use of the average rate of change that limits make precise.

On Saving Planets: OCO-2: measuring CO_2 from orbit

Sources: NOAA GML: trends in atmospheric CO_2 (https://gml.noaa.gov/ccgg/trends/) · NOAA GML: CO_2 growth rates (https://gml.noaa.gov/ccgg/trends/gr.html)

Free learning resources

- OpenStax Calculus 1, 2.2 The Limit of a Function (https://openstax.org/books/calculus-volume-1/pages/2-2-the-limit-of-a-function): Free textbook section.
- OpenStax Calculus 1, 2.3 The Limit Laws (https://openstax.org/books/calculus-volume-1/pages/2-3-the-limit-laws): Limit laws with worked examples.
- OpenStax Calculus 1, 2.5 The Precise Definition of a Limit (https://openstax.org/books/calculus-volume-1/pages/2-5-the-precise-definition-of-a-limit): ε–δ proofs.
- MIT OpenCourseWare 18.01SC Single Variable Calculus (https://ocw.mit.edu/courses/18-01sc-single-variable-calculus-fall-2010/): Free MIT lectures, notes and problem sets.

☐ **I finished this unit**

Continuity & the Derivative

Continuity means no breaks. The derivative is the limit of average rates of change: the slope of the tangent line.

Learning objectives

By the end of this unit you can:

- Test a function for continuity and use the Intermediate Value Theorem.
- Compute a derivative from the limit definition.
- Interpret a derivative as a slope, a velocity and a rate of change, with units.
- Write the equation of a tangent line.

Key concepts

- f is continuous at a if f(a) is defined, the limit exists, and they are equal.
- Intermediate Value Theorem: a continuous function on [a, b] takes every value between f(a) and f(b).
- The derivative f'(a) is the limit of the difference quotient [f(a + h) − f(a)]/h as h → 0.
- Units of f' are units of f per unit of x: meters per second, ppm per year.
- A function is not differentiable at a corner, a cusp, a vertical tangent or a break.

Key equations

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

```
f'(x)=_h0(f(x+h)-f(x))/(h)
```

Limit definition of the derivative

$$y - f(a) = f'(a)(x - a)$$

```
y-f(a)=f'(a)(x-a)
```

Tangent line at x = a

Worked examples, step by step

Example 2.1 Use the definition to find the derivative of f(x) = x².

1. [(x + h)² − x²]/h = (2xh + h²)/h = 2x + h.
2. As h → 0 this tends to 2x.

Answer: $f'(x) = 2x$.

Example 2.2 Find the tangent line to $y = x^3$ at $x = 1$.

1. $y' = 3x^2$, so the slope at $x = 1$ is 3.
2. Point: $(1, 1)$.
3. $y - 1 = 3(x - 1)$, so $y = 3x - 2$.

Answer: $y = 3x - 2$.

Example 2.3 For what a is $f(x) = x + 1$ $(x < 1)$, ax^2 $(x \geq 1)$ continuous everywhere?

1. Left limit at 1: $1 + 1 = 2$.
2. Right value and limit: $a \times 1^2 = a$.
3. Continuity needs $a = 2$.

Answer: $a = 2$.

Practice problems

Work each on paper first, then check.

1. Use the definition to differentiate $f(x) = 1/x$.

   $[1/(x + h) - 1/x]/h = -1/(x(x + h)) \to -1/x^2$.

2. A dropped stone falls $s(t) = 4.9t^2$ meters. Find its average velocity from $t = 1$ to $t = 3$ s and its instantaneous velocity at $t = 2$ s.

   Average: $(44.1 - 4.9)/2 = 19.6$ m/s. Instantaneous: $s'(t) = 9.8t$, so 19.6 m/s at $t = 2$.

3. Show that $x^3 + x - 1 = 0$ has a root between 0 and 1.

   $f(0) = -1 < 0$ and $f(1) = 1 > 0$, and f is continuous, so by the IVT it crosses 0 in between.

4. Is $f(x) = |x|$ differentiable at 0?

   No. The slope is -1 from the left and $+1$ from the right, so the limit of the difference quotient does not exist (a corner).

Common misconceptions

Myth: Continuous means differentiable.

Actually: |x| is continuous at 0 but has no derivative there. Differentiable does imply continuous, though.

Myth: The derivative is the change in y.

Actually: It is the rate of change: change in y per unit change in x, in the limit.

Myth: The tangent line touches the curve at only one point.

Actually: It can cross the curve elsewhere, and even at the point of tangency (as for $y = x^3$ at 0).

Flashcards

Continuity at a

lim f(x) as x → a equals f(a).

Intermediate Value Theorem

Continuous on [a, b] ⇒ every value between f(a) and f(b) is taken.

Difference quotient

[f(x + h) − f(x)]/h, an average rate of change.

Derivative

Limit of the difference quotient; instantaneous rate of change.

Tangent line

y = f(a) + f'(a)(x − a).

Differentiable ⇒ continuous

But not the other way round.

Self-check quiz

1. If s(t) is position in meters and t is in seconds, s'(t) has units of...

m

s

m/s

m/s^2

m/s. Units of f divided by units of x.

2. The derivative of x^2 at x = 3 is...

3

6

9

2

6. f'(x) = 2x.

3. Which function is continuous but not differentiable at 0?

x^2

$|x|$

sin x

e^x

$|x|$. It has a corner at 0.

4. The IVT requires the function to be...

differentiable

increasing

continuous on [a, b]

a polynomial

continuous on [a, b]. Continuity on the closed interval is the key hypothesis.

Try this quiz again

How this helps save the planet

The derivative turns a position record into a speed. Satellite navigation does this constantly: GRACE-FO, for example, measures how the distance between its two satellites changes, and those changes reveal shifts in water and ice on Earth.

Rates with units are how climate change is reported in practice: parts per million of CO_2 per year, millimeters of sea level per year. Reading a derivative with its units is a core science-literacy skill.

On Saving Planets: GRACE-FO · Physics I + Lab

Sources: NASA JPL: GRACE-FO (https://gracefo.jpl.nasa.gov/) · OpenStax Calculus 1, 3.1 Defining the Derivative (https://openstax.org/books/calculus-volume-1/pages/3-1-defining-the-derivative)

Free learning resources

- OpenStax Calculus 1, 2.4 Continuity (https://openstax.org/books/calculus-volume-1/pages/2-4-continuity): Continuity and the IVT.
- OpenStax Calculus 1, 3.1 Defining the Derivative (https://openstax.org/books/calculus-volume-1/pages/3-1-defining-the-derivative): The derivative from first principles.
- PhET: Calculus Grapher (https://phet.colorado.edu/en/simulations/calculus-grapher): See a function and its derivative together.

Watch the lecture

Clip 4: Physical Interpretation of Derivatives, Continued (https://ocw.mit.edu/courses/18-01sc-single-variable-calculus-fall-2010/resources/clip-4-physical-interpretation-of-derivatives-continued/) (in English)

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- [ ] **I finished this unit**

Differentiation Rules

Power, product, quotient and chain rules, plus derivatives of trigonometric, exponential and logarithmic functions, and implicit differentiation.

Learning objectives

By the end of this unit you can:

- Differentiate polynomials, products, quotients and compositions quickly and correctly.
- Differentiate sin, cos, tan, e^x and ln x.
- Use implicit differentiation to find dy/dx.
- Find higher derivatives and interpret the second derivative as acceleration.

Key concepts

- Power rule: d/dx x^n = nx^{n-1} for any real n.
- Product rule: (fg)' = f'g + fg'. Quotient rule: (f/g)' = (f'g − fg')/g^2.
- Chain rule: the derivative of f(g(x)) is f'(g(x)) · g'(x), outside derivative times inside derivative.
- e^x is its own derivative; ln x has derivative 1/x. That is why e appears in every growth and decay model.
- Implicit differentiation: differentiate both sides with respect to x, treating y as a function of x.

Key equations

$$\frac{d}{dx}x^n = nx^{n-1}$$

`d/dxx^n=nx^n-1`

Power rule

$$(fg)' = f'g + fg', \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

`(fg)'=f'g+fg', ( fg)'=(f'g-fg')/(g^2)`

Product and quotient rules

$$\frac{d}{dx}f(g(x)) = f'(g(x))\,g'(x)$$

`d/dxf(g(x))=f'(g(x)) g'(x)`

Chain rule

$$\frac{d}{dx}e^x = e^x, \quad \frac{d}{dx}\ln x = \frac{1}{x}$$

`d/dxe^x=e^x, d/dx x=1x`

Exponential and logarithm

$$\frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x$$

`d/dx x= x, d/dx x=- x`

Trigonometric derivatives

Worked examples, step by step

Example 3.1 Differentiate y = x² sin x.

1. Product rule with f = x², g = sin x.
2. y' = 2x sin x + x² cos x.

Answer: y' = 2x sin x + x² cos x.

Example 3.2 Differentiate y = (3x² + 1)⁵.

1. Outside: u⁵ → 5u⁴. Inside: u = 3x² + 1 → 6x.
2. y' = 5(3x² + 1)⁴ · 6x.

Answer: y' = 30x(3x² + 1)⁴.

Example 3.3 Find the slope of the circle x² + y² = 25 at (3, 4).

1. Differentiate: 2x + 2y y' = 0.
2. y' = −x/y = −3/4.

Answer: −3/4.

Practice problems

Work each on paper first, then check.

1. d/dx (3x⁴ − 5x² + 7)

$12x^3 - 10x$.

2. d/dx ln($x^2 + 1$)

   $2x/(x^2 + 1)$ (chain rule).

3. d/dx ($x/(x + 1)$)

   $[(x + 1) - x]/(x + 1)^2 = 1/(x + 1)^2$.

4. A quantity grows as Q(t) = 1000e^(0.02t). How fast is it growing at t = 0, and what is its relative growth rate?

   Q'(t) = 20e^(0.02t), so 20 units per unit time at t = 0. Q'/Q = 0.02, a constant 2% per unit time.

5. If $s(t) = t^3 - 6t^2$, find the velocity and acceleration at t = 1.

   $v = 3t^2 - 12t = -9$; $a = 6t - 12 = -6$.

Common misconceptions

Myth: The derivative of a product is the product of the derivatives.

Actually: Use the product rule: (fg)' = f'g + fg'.

Myth: d/dx sin(3x) = cos(3x).

Actually: Chain rule: 3 cos(3x).

Myth: d/dx $e^x = x\,e^{x-1}$.

Actually: The power rule does not apply to variable exponents. d/dx $e^x = e^x$.

Flashcards

Power rule

d/dx $x^n = nx^{n-1}$.

Product rule

(fg)' = f'g + fg'.

Quotient rule

$(f/g)' = (f'g - fg')/g^2$.

Chain rule

$[f(g(x))]' = f'(g(x))\,g'(x)$.

d/dx e^x

e^x.

d/dx ln x

$1/x$.

Self-check quiz

1. d/dx (x^5) =

$5x^4$

x^4

$5x^5$

$4x^5$

$5x^4$. Power rule.

2. d/dx sin(x^2) =

$\cos(x^2)$

$2x \cos(x^2)$

$2 \sin x \cos x$

$-2x \cos(x^2)$

$2x \cos(x^2)$. Chain rule: $\cos(x^2) \cdot 2x$.

3. d/dx ln(5x) =

$5/x$

$1/(5x)$

$1/x$

$\ln 5$

$1/x$. $5/(5x) = 1/x$.

4. If position is s(t), acceleration is...

s(t)

s'(t)

s''(t)

∫s dt

s''(t). Acceleration is the derivative of velocity.

Try this quiz again

How this helps save the planet

Growth and decay models built on e^x describe population growth, radioactive decay used to date ice cores, and compound growth in emissions. The chain rule gives their rates of change in one step.

A relative growth rate (Q'/Q) is how scientists compare changes in quantities of very different sizes, for example how fast a forest area or a greenhouse gas is changing as a percentage per year.

On Saving Planets: Deforestation hub · Earth observation

Sources: OpenStax Calculus 1, 3.9 Derivatives of Exponential and Logarithmic Functions (https://openstax.org/books/calculus-volume-1/pages/3-9-derivatives-of-exponential-and-logarithmic-functions) · NASA: carbon dioxide vital sign (https://climate.nasa.gov/vital-signs/carbon-dioxide/)

Free learning resources

- OpenStax Calculus 1, 3.3 Differentiation Rules (https://openstax.org/books/calculus-volume-1/pages/3-3-differentiation-rules): Power, product and quotient rules.
- OpenStax Calculus 1, 3.5 Derivatives of Trigonometric Functions (https://openstax.org/books/calculus-volume-1/pages/3-5-derivatives-of-trigonometric-functions): Trigonometric derivatives.
- OpenStax Calculus 1, 3.6 The Chain Rule (https://openstax.org/books/calculus-volume-1/pages/3-6-the-chain-rule): Chain rule with many examples.
- OpenStax Calculus 1, 3.8 Implicit Differentiation (https://openstax.org/books/calculus-volume-1/pages/3-8-implicit-differentiation): Implicit differentiation.

Watch the lecture

Clip 2: Introduction to General Rules for Differentiation (https://ocw.mit.edu/courses/18-01sc-single-variable-calculus-fall-2010/resources/clip-2-introduction-to-general-rules-for-differentiation/) (in English)

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☐ **I finished this unit**

Applications I: Related Rates, Approximation & Newton's Method

Use derivatives to link changing quantities, approximate hard values, evaluate tricky limits and solve equations numerically.

Learning objectives

By the end of this unit you can:

- Solve related-rates problems.
- Use linear approximation and differentials to estimate values and errors.
- Apply L'Hôpital's rule to 0/0 and ∞/∞ forms.
- Solve equations with Newton's method.

Key concepts

- Related rates: write an equation linking the quantities, differentiate with respect to time, then substitute values.
- Near x = a, f(x) ≈ f(a) + f'(a)(x − a): the tangent line is the best straight-line estimate.
- L'Hôpital's rule: for 0/0 or ∞/∞, lim f/g = lim f′/g' if the right side exists.
- Newton's method: $x_{n+1} = x_n - f(x_n)/f'(x_n)$. It usually converges very fast from a good starting guess.

Key equations

$$L(x) = f(a) + f'(a)(x - a)$$

`L(x)=f(a)+f'(a)(x-a)`

Linear approximation

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

`(f(x))/(g(x))=(f'(x))/(g'(x))`

L'Hôpital's rule (0/0 or ∞/∞)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

`x_n+1=x_n-(f(x_n))/(f'(x_n))`

Newton's method

Worked examples, step by step

Example 4.1 A spherical weather balloon's radius grows at 2 cm/s. How fast is its volume growing when r = 10 cm?

1. $V = (4/3)\pi r^3$, so $dV/dt = 4\pi r^2\, dr/dt$.
2. $dV/dt = 4\pi \times 10^2 \times 2 = 800\pi \approx 2{,}513$ cm³/s.

Answer: About 2,510 cm³/s.

Example 4.2 Estimate $\sqrt{4.1}$ with a linear approximation.

1. $f(x) = \sqrt{x}$, $a = 4$: $f(4) = 2$, $f'(4) = 1/(2\sqrt{4}) = 1/4$.
2. $L(4.1) = 2 + (1/4)(0.1) = 2.025$.

Answer: 2.025 (the true value is 2.02485…).

Example 4.3 Use Newton's method to solve $x^2 = 2$ starting from $x_0 = 1$.

1. $f(x) = x^2 - 2$, $f'(x) = 2x$, so $x_{n+1} = x_n - (x_n^2 - 2)/(2x_n)$.
2. $x_1 = 1.5$, $x_2 = 1.416667$, $x_3 = 1.414216$.

Answer: $x_3 = 1.414216$, already correct to 6 significant figures ($\sqrt{2} = 1.4142136\ldots$).

Practice problems

Work each on paper first, then check.

1. A 5 m ladder slides down a wall; its foot moves away at 0.5 m/s. How fast is the top falling when the foot is 3 m from the wall?

   $x^2 + y^2 = 25$, so $x\,x' + y\,y' = 0$. With $x = 3$, $y = 4$: $y' = -(3/4)(0.5) = -0.375$ m/s, falling at 0.375 m/s.

2. $\lim (e^x - 1)/x$ as $x \to 0$

   L'Hôpital: $e^x/1 \to 1$.

3. Newton's method for $x^3 - 2x - 5 = 0$ from $x_0 = 2$. Find x_1 and x_2.

   $f(2) = -1$, $f'(2) = 10$, so $x_1 = 2.1$. $f(2.1) = 0.061$, $f'(2.1) = 11.23$, so $x_2 = 2.0946$.

4. Use differentials to estimate the change in the area of a 10 cm radius disk if the radius grows by 0.1 cm.

   $dA = 2\pi r\,dr = 2\pi \times 10 \times 0.1 = 2\pi \approx 6.28$ cm².

Common misconceptions

Myth: L'Hôpital means differentiating the quotient f/g.

Actually: Differentiate the top and bottom separately, and only for 0/0 or ∞/∞ forms.

Myth: In related rates, plug in the numbers first, then differentiate.

Actually: Differentiate first; substituting too early turns variables into constants with zero derivative.

Myth: Newton's method always finds a root.

Actually: A bad starting point, or f' near zero, can send it off or make it cycle.

Flashcards

Related rates

Differentiate an equation linking quantities with respect to t.

Linear approximation

$f(x) \approx f(a) + f'(a)(x - a)$.

Differential

$dy = f'(x)\,dx$.

L'Hôpital's rule

For 0/0 or ∞/∞: $\lim f/g = \lim f'/g'$.

Newton's method

$x_{n+1} = x_n - f(x_n)/f'(x_n)$.

Quadratic convergence

Newton's method roughly doubles the correct digits each step near a simple root.

Self-check quiz

1. Linear approximation uses the...

secant line

tangent line

normal line

average value

tangent line. L(x) is the tangent line at a.

2. L'Hôpital's rule applies to...

any limit

0/0 and ∞/∞ forms

only polynomials

1/0 forms

0/0 and ∞/∞ forms. Only indeterminate quotients.

3. One step of Newton's method on $f(x) = x^2 - 9$ from $x_0 = 4$ gives...

3

3.125

3.5

4.5

3.125. $4 - 7/8 = 3.125$.

4. If $V = s^3$ and $ds/dt = 1$, then dV/dt at $s = 2$ is...

3

6

8

12

12. $3s^2 \, ds/dt = 12$.

Try this quiz again

How this helps save the planet

Predicting where a satellite will be means solving Kepler's equation, $M = E - e \sin E$, for E. It has no algebraic solution, so it is commonly solved numerically, often with Newton's method, as in Lab B. Accurate orbit prediction keeps Earth-observing satellites safely apart and their images correctly placed on the map.

Weather balloons, like the one in Example 4.1, carry instruments that measure the atmosphere for weather forecasts; related rates link how fast they rise and expand.

On Saving Planets: Orbits guide · Orbital period calculator

Sources: OpenStax Calculus 1, 4.9 Newton's Method (https://openstax.org/books/calculus-volume-1/pages/4-9-newtons-method) · OpenStax University Physics 1, 13.5 Kepler's Laws (https://openstax.org/books/university-physics-volume-1/pages/13-5-keplers-laws-of-planetary-motion)

Free learning resources

- OpenStax Calculus 1, 4.1 Related Rates (https://openstax.org/books/calculus-volume-1/pages/4-1-related-rates): Free textbook section.
- OpenStax Calculus 1, 4.2 Linear Approximations and Differentials (https://openstax.org/books/calculus-volume-1/pages/4-2-linear-approximations-and-differentials): Tangent-line estimates and error.
- OpenStax Calculus 1, 4.8 L'Hôpital's Rule (https://openstax.org/books/calculus-volume-1/pages/4-8-lhopitals-rule): Indeterminate forms.
- OpenStax Calculus 1, 4.9 Newton's Method (https://openstax.org/books/calculus-volume-1/pages/4-9-newtons-method): Iterative root finding.

Watch the lecture

Lecture 12: Related Rates (https://ocw.mit.edu/courses/18-01-single-variable-calculus-fall-2006/resources/lecture-12-related-rates/) (in English)

MIT 18.01 Single Variable Calculus (Fall 2006), MIT OpenCourseWare. License: CC BY-NC-SA 4.0 (https://creativecommons.org/licenses/by-nc-sa/4.0/) · Terms of use (https://ocw.mit.edu/pages/privacy-and-terms-of-use/). Linked for free study only; Saving Planets is not affiliated with or endorsed by this university.

☐ **I finished this unit**

Applications II: Extrema, Graph Shape & Optimization

Find maxima and minima, read a graph’s shape from f' and f'', and solve real design problems by optimization.

Learning objectives

By the end of this unit you can:

- Find critical points and absolute extrema on a closed interval.
- State and use the Mean Value Theorem.
- Use the first and second derivative tests and find inflection points.
- Set up and solve applied optimization problems.

Key concepts

- Critical points are where f' = 0 or f' does not exist. Extrema of a smooth function happen there or at endpoints.
- Mean Value Theorem: on a smooth interval, the instantaneous rate equals the average rate somewhere.
- f' > 0 means increasing; f'' > 0 means concave up. An inflection point is where concavity changes.
- Optimization recipe: draw it, write the quantity to optimize as a function of one variable using the constraint, differentiate, find critical points, check endpoints.

Key equations

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

`f'(c)=(f(b)-f(a))/(b-a)`

Mean Value Theorem

$$f'(c) = 0,\ f''(c) < 0 \Rightarrow \text{local max}$$

`f'(c)=0,\ f''(c)<0 max`

Second derivative test

$$C_p(a) = 4a(1 - a)^2$$

`C_p(a)=4a(1-a)^2`

Wind-turbine power coefficient vs. axial induction a (actuator-disk model)

Worked examples, step by step

Example 5.1 Find the local extrema of $f(x) = x^3 - 3x$.

1. $f'(x) = 3x^2 - 3 = 0$ at $x = \pm 1$.
2. $f''(x) = 6x$: $f''(-1) = -6 < 0$ (max), $f''(1) = 6 > 0$ (min).
3. $f(-1) = 2$, $f(1) = -2$.

Answer: Local max 2 at $x = -1$; local min -2 at $x = 1$.

Example 5.2 A closed cylindrical can must hold 500 cm^3. What radius uses the least metal?

1. Constraint: $\pi r^2 h = 500$, so $h = 500/(\pi r^2)$.
2. Area $A = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 1000/r$.
3. $A' = 4\pi r - 1000/r^2 = 0$ gives $r^3 = 500/(2\pi)$, $r = 4.30$ cm.
4. Then $h = 8.60$ cm $= 2r$, and $A = 349$ cm^2.

Answer: $r = 4.30$ cm, $h = 8.60$ cm (height equals diameter).

Example 5.3 In the simplest model of a wind turbine, the fraction of wind power captured is $Cp = 4a(1 - a)^2$, where a is how much the rotor slows the wind. Find the maximum.

1. $Cp' = 4(1 - a)^2 - 8a(1 - a) = 4(1 - a)(1 - 3a)$.
2. Critical points $a = 1$ and $a = 1/3$; $a = 1$ gives $Cp = 0$.
3. $Cp(1/3) = 4 \times (1/3) \times (2/3)^2 = 16/27 = 0.593$.

Answer: 59.3%: the Betz limit, the theoretical maximum for any wind turbine.

Practice problems

Work each on paper first, then check.

1. A farmer has 100 m of fence for three sides of a rectangle along a river. What is the largest area?

   $A = x(100 - 2x)$; $A' = 100 - 4x = 0$ at $x = 25$. Area $= 25 \times 50 = 1{,}250$ m^2.

2. Find the absolute max and min of $f(x) = x^2 - 4x$ on $[0, 5]$.

   Critical point $x = 2$: $f = -4$. Endpoints: $f(0) = 0$, $f(5) = 5$. Max 5, min -4.

3. Verify the MVT for $f(x) = x^2$ on $[1, 3]$.

   Average rate $(9 - 1)/2 = 4$; $f'(c) = 2c = 4$ at $c = 2$, inside $[1, 3]$.

4. Where is $f(x) = x^3 - 6x^2$ concave up?

   $f'' = 6x - 12 > 0$ when $x > 2$; inflection point at $x = 2$.

Common misconceptions

Myth: $f'(c) = 0$ always means a max or min.

Actually: x^3 has $f'(0) = 0$ but no extremum there.

Myth: The absolute max is always at a critical point.

Actually: On a closed interval it can be at an endpoint. Always check them.

Myth: Concave up means increasing.

Actually: Concavity is about f''; a function can be concave up and decreasing, like e^{-x}.

Flashcards

Critical point

$f' = 0$ or undefined.

Extreme Value Theorem

Continuous on $[a, b] \Rightarrow$ absolute max and min exist.

Mean Value Theorem

$f'(c) = [f(b) - f(a)]/(b - a)$ for some c.

First derivative test

Sign change of f' from + to − is a max; − to + is a min.

Concavity

$f'' > 0$ up, $f'' < 0$ down.

Betz limit

Max $C_p = 16/27 \approx 59.3\%$.

Self-check quiz

1. If $f'(c) = 0$ and $f''(c) > 0$, then at c f has a...

local max

local min

inflection point

no conclusion

local min. Concave up at a flat point: a valley.

2. The Betz limit for wind turbines is about...

33%

50%

59%

100%

59%. 16/27 ≈ 0.593, from maximizing $4a(1 - a)^2$.

3. The MVT guarantees a point where...

f = 0

f' = 0 always

f' equals the average rate of change

f'' = 0

f' equals the average rate of change. Instantaneous rate = average rate.

4. An inflection point is where...

f' = 0

f changes sign

concavity changes

f is undefined

concavity changes. f'' changes sign.

Try this quiz again

How this helps save the planet

Optimization is engineering. The Betz limit in Example 5.3 comes from one derivative set to zero, and it tells wind engineers the most any turbine could ever extract from the wind.

Mission designers optimize too: a Hohmann transfer is usually the most fuel-efficient two-burn route between two circular orbits, and less propellant means smaller, cheaper launches for climate and weather satellites.

Using the least material for a job, like the can in Example 5.2, cuts the energy and resources that go into making it.

On Saving Planets: Hohmann transfer calculator · Physics I, Unit 4: wind power

Sources: U.S. DOE: How do wind turbines work? (https://www.energy.gov/eere/wind/how-do-wind-turbines-work) · OpenStax Calculus 1, 4.7 Applied Optimization Problems (https://openstax.org/books/calculus-volume-1/pages/4-7-applied-optimization-problems)

Free learning resources

- OpenStax Calculus 1, 4.3 Maxima and Minima (https://openstax.org/books/calculus-volume-1/pages/4-3-maxima-and-minima): Extrema and critical points.
- OpenStax Calculus 1, 4.5 Derivatives and the Shape of a Graph (https://openstax.org/books/calculus-volume-1/pages/4-5-derivatives-and-the-shape-of-a-graph): First and second derivative tests.
- OpenStax Calculus 1, 4.7 Applied Optimization Problems (https://openstax.org/books/calculus-volume-1/pages/4-7-applied-optimization-problems): Many solved design problems.

☐ **I finished this unit**

UNIT 6 · WEEKS 12–14

Integrals & the Fundamental Theorem of Calculus

Integrals add up tiny pieces: area, distance, energy. The Fundamental Theorem links them to derivatives.

Learning objectives

By the end of this unit you can:

- Approximate areas with Riemann sums and the trapezoid rule.
- Define the definite integral and use its properties.
- Use both parts of the Fundamental Theorem of Calculus.
- Find antiderivatives, including by substitution, and compute net change, average value and areas between curves.

Key concepts

- A Riemann sum adds $f(x_i)\Delta x$ over small slices; the definite integral is its limit as the slices shrink.
- FTC part 1: $d/dx \int_a^x f(t)\,dt = f(x)$. Integration and differentiation undo each other.
- FTC part 2: $\int_a^b f(x)\,dx = F(b) - F(a)$ for any antiderivative F.
- Net change theorem: integrating a rate gives the total change. Integrating power gives energy; integrating velocity gives displacement.
- Substitution is the chain rule in reverse.

Key equations

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

```
_a^bf(x) dx=_n∞_i=1^nf(x_i^*)Δ x
```

Definite integral

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

```
d/dx_a^xf(t) dt=f(x)
```

FTC part 1

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

```
_a^bf(x) dx=F(b)-F(a)
```

FTC part 2

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C\ (n \neq -1), \quad \int \frac{1}{x}\,dx = \ln|x| + C$$

```
x^n dx=^n+1n+1+C\ (n-1), 1x dx=|x|+C
```

Basic antiderivatives

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

```
f_avg=1b-a_a^bf(x) dx
```

Average value

Worked examples, step by step

Example 6.1 Find $\int_0^2 x^2\,dx$.

1. Antiderivative $x^3/3$.
2. $8/3 - 0 = 8/3$.

Answer: $8/3 \approx 2.667$.

Example 6.2 Find $\int_0^1 2x(x^2+1)^3\,dx$.

1. Let $u = x^2 + 1$, $du = 2x\,dx$; u goes from 1 to 2.
2. $\int_1^2 u^3\,du = [u^4/4]_1^2 = (16 - 1)/4$.

Answer: $15/4 = 3.75$.

Example 6.3 A solar array's output on a clear day is modeled as $P(t) = 5\sin(\pi t/12)$ kW for $0 \le t \le 12$ hours after sunrise. How much energy does it make?

1. Energy $= \int_0^{12} 5\sin(\pi t/12)\,dt$.
2. Antiderivative: $-(60/\pi)\cos(\pi t/12)$.
3. $-(60/\pi)(\cos\pi - \cos 0) = 120/\pi$.

Answer: $120/\pi \approx 38.2$ kWh (a simplified model, not a real array).

Practice problems

Work each on paper first, then check.

1. $\int_0^{\pi} \sin x \, dx$

   $[-\cos x]_0^{\pi} = 1 + 1 = 2$.

2. $\frac{d}{dx} \int_0^{x} \cos(t^2) \, dt$

   $\cos(x^2)$, by FTC part 1.

3. A rocket's upward velocity is $v(t) = 9.8t$ m/s for 3 s. How far does it rise?

   $\int_0^3 9.8t \, dt = 4.9 \times 9 = 44.1$ m.

4. Find the area between $y = x$ and $y = x^2$ on $[0, 1]$.

   $\int_0^1 (x - x^2) \, dx = 1/2 - 1/3 = 1/6$.

5. Find the average value of $f(x) = x^2$ on $[0, 3]$.

   $(1/3) \int_0^3 x^2 \, dx = (1/3)(9) = 3$.

Common misconceptions

Myth: An integral is always an area, so it is always positive.

Actually: It is a signed area: parts below the axis count as negative. $\int_0^{2\pi} \sin x \, dx = 0$.

Myth: Forgetting + C does not matter.

Actually: For indefinite integrals the constant is part of the answer; it matters whenever an initial condition is given.

Myth: In substitution, keep the old limits.

Actually: Change the limits to u-values, or substitute back before evaluating.

Flashcards

Riemann sum

$\Sigma f(x_i^*) \Delta x$; an area estimate.

Definite integral

Limit of Riemann sums; signed area.

FTC part 1

$\frac{d}{dx} \int_a^x f = f(x)$.

FTC part 2

$\int_a^b f = F(b) - F(a)$.

Net change

Integral of a rate = total change.

Substitution

u = g(x), du = g'(x) dx: the chain rule in reverse.

Self-check quiz

1. $\int_1^3 2x\,dx =$

4

8

9

6

8. $[x^2]_1^3 = 9 - 1 = 8$.

2. Integrating a power curve (kW) over time (h) gives...

kW

kWh of energy

kW/h

average power

kWh of energy. Net change theorem: the integral of a rate is a total.

3. $\frac{d}{dx}\int_2^x t^3\,dt =$

x^3

$3x^2$

$x^4/4$

$x^3 - 8$

x^3. FTC part 1.

4. $\int 1/x\,dx =$

$x^0/0$

$\ln|x| + C$

$-1/x^2 + C$

$e^x + C$

$\ln|x| + C$. The $n = -1$ exception to the power rule.

Try this quiz again

How this helps save the planet

Energy is the integral of power over time. Grid planners add up hour-by-hour solar and wind output this way to know how much clean energy they will get and how much storage they need.

Satellite instruments measure rates, such as the flow of sunlight or of water, and integrals turn those rates into totals: how much energy Earth absorbs, or how much water a river basin gains or loses over a season.

The numerical integration in Lab D is exactly how real data, which comes as separate measurements, gets integrated.

On Saving Planets: GRACE-FO · Calculators

Sources: OpenStax Calculus 1, 5.3 The Fundamental Theorem of Calculus (https://openstax.org/books/calculus-volume-1/pages/5-3-the-fundamental-theorem-of-calculus) · U.S. DOE: How do wind turbines work? (https://www.energy.gov/eere/wind/how-do-wind-turbines-work)

Free learning resources

- OpenStax Calculus 1, 5.1 Approximating Areas (https://openstax.org/books/calculus-volume-1/pages/5-1-approximating-areas): Riemann sums.
- OpenStax Calculus 1, 5.2 The Definite Integral (https://openstax.org/books/calculus-volume-1/pages/5-2-the-definite-integral): Definition and properties.
- OpenStax Calculus 1, 5.3 The Fundamental Theorem of Calculus (https://openstax.org/books/calculus-volume-1/pages/5-3-the-fundamental-theorem-of-calculus): Both parts with proofs.
- OpenStax Calculus 1, 5.5 Substitution (https://openstax.org/books/calculus-volume-1/pages/5-5-substitution): u-substitution.
- OpenStax Calculus 1, 6.1 Areas between Curves (https://openstax.org/books/calculus-volume-1/pages/6-1-areas-between-curves): Areas between curves.

Watch the lecture

Clip 1: Proof of the Second Fundamental Theorem of Calculus (https://ocw.mit.edu/courses/18-01sc-single-variable-calculus-fall-2010/resources/clip-1-proof-of-the-second-fundamental-theorem-of-calculus/) (in English)

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- [ ] **I finished this unit**

Problem Sets

These practice problems live inside each unit. Work on paper first, then open the answer.

1. Functions & Limits
   5 problems
2. Continuity & the Derivative
   4 problems
3. Differentiation Rules
   5 problems
4. Applications I: Related Rates, Approximation & Newton's Method
   4 problems
5. Applications II: Extrema, Graph Shape & Optimization
   4 problems
6. Integrals & the Fundamental Theorem of Calculus
   5 problems

Computer labs

Calculus has no wet lab, so this course has four short computer labs instead. Each one applies a unit to real-world style data with a spreadsheet: finite differences, Newton's method, optimization and numerical integration.

These are the same numerical methods engineers and climate scientists use every day. The sample data in each lab is for practice; each lab links to the real data source.

Safety reference: NOAA Global Monitoring Laboratory: trends in CO_2 (https://gml.noaa.gov/ccgg/trends/)

Lab A: Rates of change from data

WEEK 4

Goal: Estimate a rate of change and its acceleration from a table of measurements, the way climate scientists treat a CO_2 record.

$$f'(t_i) \approx \frac{f(t_{i+1}) - f(t_{i-1})}{t_{i+1} - t_{i-1}}, \quad f''(t_i) \approx \frac{f(t_{i+1}) - 2f(t_i) + f(t_{i-1})}{h^2}$$

Central differences for the first and second derivative

Procedure

1. Enter the sample table in a spreadsheet.
2. Compute forward differences (rate over each year) and central differences.
3. Compute the second difference: is the rate itself rising?
4. Download real annual means from NOAA and repeat; compare with NOAA's published growth rates.

Data table

Year t	C (ppm)
0	400.0
1	402.4
2	404.9
3	407.5
4	410.2

Sample values for practice, not real measurements. Use NOAA's data for the real thing.

Sample calculation

1. Central difference at t = 2: (407.5 − 402.4)/2 = 2.55 ppm per year.
2. Second difference at t = 2: 407.5 − 2(404.9) + 402.4 = 0.1 ppm per year².
3. A positive second difference means the growth rate is itself increasing (in this sample).

> **Safety**
>
> - Keep a copy of the raw data untouched; do calculations in new columns.

Home or virtual alternative

- Everything here is done on a computer or even by hand.

NOAA GML: CO_2 growth rates (https://gml.noaa.gov/ccgg/trends/gr.html)

Planet link: Rates of change are how CO_2 trends are reported and compared. NOAA GML: trends in atmospheric CO_2 (https://gml.noaa.gov/ccgg/trends/)

☐ **I completed this lab (wet or virtual)**

Lab B: Newton’s method and Kepler’s equation

WEEK 9

Goal: Solve Kepler’s equation M = E − e sin E for a satellite’s eccentric anomaly E, the key step in predicting where it is.

$$E_{n+1} = E_n - \frac{E_n - e\sin E_n - M}{1 - e\cos E_n}$$

```
E_n+1=E_n-(E_n-e E_n-M)/(1-e E_n)
```

Newton’s method for Kepler’s equation

Procedure

1. Set M = 1.0 rad and e = 0.1 in a spreadsheet; start with E_0 = M.
2. Fill down the Newton formula for four rows.
3. Watch how many digits stop changing each step.
4. Try e = 0.9 and E_0 = M: how many steps does it need now?

Data table

Step n	E_n (rad)
0	1.000000
1	1.088953
2	1.088598
3	1.088598

Computed values (M = 1.0 rad, e = 0.1).

Sample calculation

1. f(E) = E − 0.1 sin E − 1; f'(E) = 1 − 0.1 cos E.
2. From E_0 = 1: E_1 = 1.088953, E_2 = 1.088598, after which the first 7 digits stop changing.
3. Check: 1.088598 − 0.1 sin(1.088598) = 1.000000.

> **Safety**
>
> - None beyond normal computer use.

Home or virtual alternative

- A spreadsheet or Desmos is enough.

OpenStax Calculus 1, 4.9 Newton's Method (https://openstax.org/books/calculus-volume-1/pages/4-9-newtons-method)

Planet link: Orbit prediction keeps Earth-observing satellites safe and their data correctly placed.

- [ ] **I completed this lab (wet or virtual)**

Lab C: Optimizing a wind turbine

WEEK 11

Goal: Confirm the Betz limit numerically and by calculus.

$$C_p(a) = 4a(1-a)^2, \quad C_p'(a) = 4(1-a)(1-3a)$$

`C_p(a)=4a(1-a)^2, C_p'(a)=4(1-a)(1-3a)`

Power coefficient and its derivative

Procedure

1. Tabulate Cp for a = 0 to 0.5 in steps of 0.05 and graph it.
2. Find the highest value in the table.
3. Solve Cp'(a) = 0 by hand and compare.

Data table

a	Cp
0.20	0.512
0.30	0.588
1/3	0.593
0.40	0.576

Computed values from the model.

Sample calculation

1. Cp(0.3) = 4 × 0.3 × 0.7² = 0.588.
2. Cp(1/3) = 16/27 = 0.593, the maximum.
3. Real turbines capture less because of drag, wake rotation and tip losses.

> **Safety**
>
> - None beyond normal computer use.

Home or virtual alternative

- A spreadsheet or Desmos is enough.

U.S. DOE: How do wind turbines work? (https://www.energy.gov/eere/wind/how-do-wind-turbines-work)

Planet link: Knowing the ceiling tells engineers how much improvement is still possible.

- [ ] **I completed this lab (wet or virtual)**

Lab D: Numerical integration of solar power

WEEK 13

Goal: Integrate a power curve with the midpoint and trapezoid rules and compare with the exact answer.

$$T_n = \frac{\Delta t}{2}\left[f_0 + 2f_1 + \cdots + 2f_{n-1} + f_n\right]$$

```
T_n=Δ t2[f_0+2f_1++2f_n-1+f_n]
```

Trapezoid rule

Procedure

1. Model P(t) = 5 sin(πt/12) kW on 0 ≤ t ≤ 12 h.
2. With n = 6 slices (Δt = 2 h), compute the midpoint and trapezoid sums.
3. Compare with the exact 120/π kWh. Double n and watch the errors shrink.

Data table

Method (n = 6)	Energy (kWh)
Midpoint	38.64
Trapezoid	37.32
Exact (FTC)	38.20

Model values, not data from a real array.

Sample calculation

1. Trapezoid: 2 × [0 + 2(2.5 + 4.330 + 5 + 4.330 + 2.5)]/2 = 37.32 kWh.
2. Midpoint: 2 × Σ P at t = 1, 3, ..., 11 = 38.64 kWh.
3. Errors: midpoint +0.44, trapezoid −0.88. The midpoint error is about half the trapezoid error, with opposite sign, as theory predicts.

> **Safety**
>
> - None beyond normal computer use.

Home or virtual alternative

- A spreadsheet or Desmos is enough.

OpenStax Calculus 1, 5.1 Approximating Areas (https://openstax.org/books/calculus-volume-1/pages/5-1-approximating-areas)

Planet link: This is how real, hourly energy data is added up to plan clean power and storage.

- [ ] **I completed this lab (wet or virtual)**

Exam prep practice sets

Time yourself: about 10 minutes per problem. Then compare with the full solutions.

Midterm practice set (Units 1–3)

1. Find $\lim (x - 4)/(\sqrt{x} - 2)$ as $x \to 4$.

    1. Multiply by $\sqrt{x} + 2$: $(x - 4)(\sqrt{x} + 2)/(x - 4) = \sqrt{x} + 2$.
    2. Substitute $x = 4$: $2 + 2$.

    Answer: 4.

2. Differentiate $y = x^3 e^x$ and evaluate at $x = 1$.

    1. Product rule: $y' = 3x^2 e^x + x^3 e^x = (3x^2 + x^3)e^x$.
    2. At $x = 1$: $4e$.

    Answer: $4e \approx 10.87$.

3. Find the tangent line to $y = \ln x$ at $x = 1$.

    1. $y(1) = 0$, $y' = 1/x = 1$.
    2. $y - 0 = 1(x - 1)$.

    Answer: $y = x - 1$.

4. Differentiate $y = \sin(3x^2)$.

    1. Chain rule: $\cos(3x^2) \cdot 6x$.

    Answer: $y' = 6x \cos(3x^2)$.

Final practice set (all units)

1. Water flows into a cylindrical tank of radius 2 m at 0.5 m^3/min. How fast does the level rise?

    1. $V = \pi r^2 h = 4\pi h$, so $dV/dt = 4\pi \, dh/dt$.
    2. $dh/dt = 0.5/(4\pi)$.

Answer: About 0.0398 m/min (4.0 cm per minute).

2. Find $\int_1^e (1/x)\,dx$.

    1. $[\ln x]_1^e = 1 - 0$.

    Answer: 1.

3. Find $\int_0^4 \sqrt{x}\,dx$.

    1. Antiderivative $(2/3)x^{(3/2)}$.
    2. $(2/3)(8) = 16/3$.

    Answer: $16/3 \approx 5.33$.

4. A rectangle has perimeter 40 m. What dimensions maximize its area?

    1. $A = x(20 - x)$; $A' = 20 - 2x = 0$ at $x = 10$.

    Answer: A 10 m × 10 m square, area 100 m².

5. Use L'Hôpital to find $\lim x^2/e^x$ as $x \to \infty$.

    1. ∞/∞: $\to 2x/e^x$, still ∞/∞.
    2. $\to 2/e^x \to 0$.

    Answer: 0: exponentials outgrow any power.

Formula sheet

$$f'(x) = \lim_{h \to 0} [f(x+h) - f(x)]/h \qquad (x^n)' = nx^{n-1}$$

$$(fg)' = f'g + fg' \qquad (f/g)' = (f'g - fg')/g^2$$

$$[f(g(x))]' = f'(g(x))g'(x) \qquad (e^x)' = e^x,\ (\ln x)' = 1/x$$

$$(\sin x)' = \cos x,\ (\cos x)' = -\sin x,\ (\tan x)' = \sec^2 \qquad L(x) = f(a) + f'(a)(x - a)$$

$$x_{n+1} = x_n - f(x_n)/f'(x_n) \qquad f'(c) = [f(b) - f(a)]/(b - a)$$

$$\int x^n\,dx = x^{n+1}/(n+1) + C \qquad \int e^x\,dx = e^x + C,\ \int dx/x = \ln|x| + C$$

$$\int \sin x\,dx = -\cos x + C,\ \int \cos x\,dx = \sin x + C$$

$$\int_a^b f\,dx = F(b) - F(a)$$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$f_{avg} = \frac{1}{b-a}\int_a^b f\,dx$$

Constants

Constant	Value
e	2.718281828...
π	3.141592654...
ln 2	0.693147...
1 radian	57.2958°

Source: NIST: Fundamental physical constants (https://physics.nist.gov/cuu/Constants/)

Graphing and data tools

Use these free tools and data sources alongside the course.

- Desmos graphing calculator (https://www.desmos.com/calculator): Graph functions, sliders and tangent lines in the browser.
- PhET: Calculus Grapher (https://phet.colorado.edu/en/simulations/calculus-grapher): Draw a function and see its derivative and integral.
- NOAA: trends in atmospheric CO_2 (https://gml.noaa.gov/ccgg/trends/): Real Mauna Loa and global CO_2 data to practice rates of change.

Glossary

Antiderivative
A function F with F' = f.

Asymptote
A line a graph approaches far out or near a blow-up.

Chain rule
Derivative of a composition: outside' × inside'.

Concavity
Direction a graph bends; sign of f''.

Continuity
No breaks: the limit equals the function value.

Critical point
Where f' is zero or undefined.

Definite integral
Signed area under a curve between two limits.

Derivative
Instantaneous rate of change; slope of the tangent line.

Differential
dy = f'(x) dx.

Fundamental Theorem of Calculus
Links derivatives and integrals; $\int_a^b f = F(b) - F(a)$.

Indeterminate form
0/0 or ∞/∞; needs more work.

Inflection point
Where concavity changes.

Limit
The value a function approaches.

Linear approximation
Tangent-line estimate.

Mean Value Theorem
Instantaneous rate equals average rate somewhere.

Newton's method
Iterative root finder using tangent lines.

Optimization
Finding the best value of a quantity under constraints.

Riemann sum
Sum of rectangle areas approximating an integral.

Substitution
Integration technique reversing the chain rule.

Trapezoid rule
Numerical integration with trapezoids.

Self-assessment rubric

No grade or credit is given. Use this to judge honestly where you are in each unit.

Level	What it looks like
4 · Mastery	Solves unseen problems, justifies each step with a rule or theorem, and checks answers with a graph or estimate.

Level	What it looks like
3 · Proficient	Solves the practice problems correctly with occasional algebra slips.
2 · Developing	Follows worked examples but struggles to choose a method, or misapplies the chain rule.
1 · Beginning	Memorizes formulas without knowing when they apply; needs to revisit the unit.

Suggested weighting if you want a score

Part	Weight
Practice problems	30%
Computer labs	15%
Midterm practice set	25%
Final practice set	30%

Aim for level 3 or better in every unit before moving to the next course.

Related courses

- Physics I + Lab PHYS 110 + PHYS 110L
- Calculus II MATH 111
- General Chemistry I + Lab CHEM 110 + CHEM 110L
- All courses