Last updated: November 1, 2026

🗺️ Four-year roadmap · Year 1, Spring

About this course

This free course has 7 units and 4 labs. Those are the real counts, not a longer syllabus cut short.

A check is one number or code result on a course page that a script recomputes before the page is published. The printed number and the recomputed number have to match.

> This is a free self-study course outline written by Saving Planets. It is not a university course, it carries no academic credit, and no school is involved.
>
> Its scope follows second-semester college calculus, modeled on the topics and depth of open materials such as OpenStax Calculus Volume 2 and MIT OpenCourseWare 18.01SC (with parametric and polar ideas that lead into 18.02SC Multivariable Calculus), which we link to throughout. Those organizations do not endorse this page.
>
> What we add: every unit connects to real water, energy, ecology and spaceflight problems, every number on the page is re-computed by a checking script before it goes live, and four computer labs put the methods to work on data.
>
> Work through it in order. Each unit has goals, key ideas, equations, step-by-step examples, practice with full answers, common mistakes, flashcards and a quiz. Your progress is saved only in this browser.

How this course helps save the planet

Calculus II is where calculus starts modeling the world: volumes of water in reservoirs, the work to pump it, how a pollutant flushes out of a lake, how a fish population recovers, and the elliptical orbits of satellites that watch it all.

Each unit ends with a short, sourced section showing how its ideas help both people and nature.

For instructors

You can teach this course, or let a student work through it alone. It is free. Saving Planets does not give a grade or school credit.

- This page is the whole course: 7 units and 4 labs (MATH 111).
- Plan on 15 weeks, about 6 to 9 hours a week, including the lab weeks.
- Follow the 15-week schedule. Read the unit, do the practice on paper, then open the answer. In a lab week, do the lab or the home version.
- A check is one number or code result on the page that a script recomputes before we publish. The number you see and the recomputed number have to match.
- Use Print this page, or download the PDF study guide. The formula sheet is near the end.
- Ask: how does this unit help people and the planet? Each unit has a green box with a sourced fact.
- Copy the teacher guide template and write in your class name, the time you have, and the questions you will ask. Open the teacher guide template

Before you start

Prerequisites

- Calculus I: limits, derivatives, the chain rule, basic integrals and substitution, the Fundamental Theorem.
- Trigonometric identities, especially $\sin^2 x + \cos^2 x = 1$ and the double-angle formulas.
- Exponentials and logarithms.

How to study

- Read the unit, then close it and try the worked examples yourself before checking each step.
- Do every practice problem on paper before opening the answer.
- For integrals, check your answer by differentiating it.
- For series, decide which test fits before computing anything.
- Sketch every region before setting up a volume or area integral.
- Retake the quizzes a few days later. Spaced practice beats one long session.

What you need

- A computer or tablet with a spreadsheet.
- A free graphing calculator such as Desmos.
- Optional: Python with NumPy.

Suggested 15-week schedule

About 6 to 9 hours a week: reading, practice problems and, in lab weeks, the lab.

Week	Unit	Topics	Reading
1	Techniques of Integration	Integration by parts	OpenStax 3.1

Week	Unit	Topics	Reading
2	Techniques of Integration	Trigonometric integrals and substitution	OpenStax 3.2–3.3
3	Techniques of Integration	Partial fractions; numerical integration	OpenStax 3.4–3.6
4	Applications: Volume, Arc Length, Work & Center of Mass	Areas, volumes by slicing, disks, washers, shells	OpenStax 2.1–2.3
5	Applications: Volume, Arc Length, Work & Center of Mass	Arc length, work, fluid force, centers of mass; Lab B	OpenStax 2.4–2.6
6	Improper Integrals	Improper integrals and comparison; midterm practice	OpenStax 3.7
7	Introduction to Differential Equations	Differential equations, direction fields, Euler's method	OpenStax 4.1–4.2
8	Introduction to Differential Equations	Separable and logistic equations; Lab A	OpenStax 4.3–4.4
9	Sequences & Series	Sequences; infinite series; geometric series	OpenStax 5.1–5.2
10	Sequences & Series	Divergence, integral and comparison tests	OpenStax 5.3–5.4
11	Sequences & Series	Alternating series; ratio and root tests	OpenStax 5.5–5.6
12	Power Series & Taylor Series	Power series and their properties	OpenStax 6.1–6.2
13	Power Series & Taylor Series	Taylor and Maclaurin series; Lab C	OpenStax 6.3–6.4
14	Parametric Equations & Polar Coordinates	Parametric equations; polar coordinates and areas; Lab D	OpenStax 7.1–7.4
15	Review	Review and final practice set	All units

Your progress

Tick units and labs as you finish them, and take the quizzes. Progress and quiz scores are saved only in this browser, never sent to us.

Units

Techniques of Integration

Integration by parts, trigonometric integrals and substitutions, partial fractions, and numerical integration when no formula exists.

Learning objectives

By the end of this unit you can:

- Integrate by parts, choosing u and dv well.
- Evaluate trigonometric integrals and use trigonometric substitution.
- Decompose rational functions into partial fractions.
- Approximate integrals with the trapezoid rule and Simpson’s rule, and estimate the error.

Key concepts

- Integration by parts is the product rule in reverse: ∫u dv = uv − ∫v du. Choose u to get simpler when differentiated (logs, polynomials).
- Powers of sine and cosine yield to identities such as sin²x = (1 − cos 2x)/2.
- For √(a² − x²), substitute x = a sin θ; for √(a² + x²), x = a tan θ.
- A proper rational function splits into simple fractions whose integrals are logs and arctangents.
- Many integrals, such as ∫e^(−x²) dx, have no elementary antiderivative. Simpson’s rule is often accurate to many digits with few points.

Key equations

$$\int u\,dv = uv - \int v\,du$$

`u dv=uv- v du`

Integration by parts

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$

`^2x=12(1-2x)`

Power-reduction identity

$$\frac{1}{x^2 - 1} = \frac{1/2}{x - 1} - \frac{1/2}{x + 1}$$

`(1)/(x^2-1)=(1/2)/(x-1)-(1/2)/(x+1)`

A partial-fraction decomposition

$$S_n = \frac{\Delta x}{3}\left[f_0 + 4f_1 + 2f_2 + \cdots + 4f_{n-1} + f_n\right]$$

`S_n=(Δ x)/(3)[f_0+4f_1+2f_2++4f_n-1+f_n]`

Simpson's rule (n even)

Worked examples, step by step

Example 1.1 Find $\int_0^1 x\,e^x\,dx$.

1. $u = x$, $dv = e^x\,dx$, so $du = dx$, $v = e^x$.
2. $\int x\,e^x\,dx = x\,e^x - \int e^x\,dx = (x - 1)e^x$.
3. From 0 to 1: $0 - (-1) = 1$.

Answer: 1.

Example 1.2 Find $\int_2^3 dx/(x^2 - 1)$.

1. Partial fractions: $\tfrac{1}{2}[1/(x - 1) - 1/(x + 1)]$.
2. Antiderivative $\tfrac{1}{2}\ln|(x - 1)/(x + 1)|$.
3. $\tfrac{1}{2}[\ln(2/4) - \ln(1/3)] = \tfrac{1}{2}\ln(3/2)$.

Answer: $\tfrac{1}{2}\ln 1.5 \approx 0.2027$.

Example 1.3 Estimate $\int_0^1 e^{\wedge}(-x^2)\,dx$ with Simpson's rule, $n = 4$.

1. $\Delta x = 0.25$; $f = 1, 0.93941, 0.77880, 0.56978, 0.36788$.
2. $S_4 = (0.25/3)[1 + 4(0.93941) + 2(0.77880) + 4(0.56978) + 0.36788]$.
3. $S_4 = 0.74686$.

Answer: 0.74686, versus the true 0.74682: an error of only 0.00003 from five points.

Practice problems

Work each on paper first, then check.

1. $\int_0^{\wedge}\pi\, x \sin x\,dx$

   Parts: $[-x\cos x + \sin x]_0^{\wedge}\pi = \pi$.

2. $\int_1^e x \ln x\,dx$

Parts with $u = \ln x$: $[x^2 \ln x/2 - x^2/4]_1^e = (e^2 + 1)/4 \approx 2.097$.

3. $\int_0^{\pi/2} \sin^2 x\, dx$

   Power reduction: $\frac{1}{2}[x - \sin 2x/2]_0^{\pi/2} = \pi/4$.

4. $\int_0^1 \sqrt{1 - x^2}\, dx$

   $x = \sin\theta$ gives $\int_0^{\pi/2} \cos^2\theta\, d\theta = \pi/4$ (a quarter of the unit disk).

5. $\int_0^1 dx/(1 + x^2)$

   $[\arctan x]_0^1 = \pi/4$.

Common misconceptions

Myth: Always let u be the first factor you see.

Actually: Choose u to simplify when differentiated: logs and polynomials first, exponentials and trig as dv.

Myth: Partial fractions works on any rational function.

Actually: Divide first if the top's degree is not less than the bottom's.

Myth: If I can't find an antiderivative, the integral has no value.

Actually: It still has a value; estimate it numerically.

Flashcards

Integration by parts

$\int u\, dv = uv - \int v\, du$.

Trig substitution

$\sqrt{a^2 - x^2}$: $x = a \sin\theta$.

Partial fractions

Split a rational function into simple fractions.

Trapezoid rule

Average of left and right sums; error $\propto \Delta x^2$.

Simpson's rule

Parabolic fit; error $\propto \Delta x^4$.

Non-elementary integral

No formula antiderivative, e.g. $\int e^{-x^2}\, dx$.

Self-check quiz

1. For $\int x^2 e^x\, dx$, a good first choice of u is...

e^x

x^2

$x^2 e^x$

dx

x^2. x^2 gets simpler when differentiated.

2. $\sqrt{(9 - x^2)}$ suggests the substitution...

$x = 3 \tan \theta$

$x = 3 \sin \theta$

$x = 3 \sec \theta$

$u = 9 - x^2$

$x = 3 \sin \theta$. $9 - 9 \sin^2\theta = 9 \cos^2\theta$.

3. Simpson's rule fits the curve with...

rectangles

trapezoids

parabolas

cubics only

parabolas. Parabolas through each set of three points.

4. $\int \ln x \, dx =$

$1/x + C$

$x \ln x - x + C$

$\ln^2 x/2 + C$

$x \ln x + C$

x ln x − x + C. Parts with u = ln x, dv = dx.

Try this quiz again

How this helps save the planet

The bell-curve integral in Example 1.3 has no formula answer, yet it sits behind every error bar on a climate graph and every uncertainty in a satellite measurement. Numerical integration is how it, and most integrals from real data, actually get computed.

Methods like Simpson's rule are how irregular measurements, such as river depths across a channel, become totals like flow or volume.

On Saving Planets: Calculus I, Lab D: numerical integration · Rivers hub

Sources: OpenStax Calculus 2, 3.6 Numerical Integration (https://openstax.org/books/calculus-volume-2/pages/3-6-numerical-integration) · OpenStax Calculus 2, 3.1 Integration by Parts (https://openstax.org/books/calculus-volume-2/pages/3-1-integration-by-parts)

Free learning resources

- OpenStax Calculus 2, 3.1 Integration by Parts (https://openstax.org/books/calculus-volume-2/pages/3-1-integration-by-parts): Free textbook section.
- OpenStax Calculus 2, 3.2 Trigonometric Integrals (https://openstax.org/books/calculus-volume-2/pages/3-2-trigonometric-integrals): Powers of sine and cosine.
- OpenStax Calculus 2, 3.3 Trigonometric Substitution (https://openstax.org/books/calculus-volume-2/pages/3-3-trigonometric-substitution): Radical integrands.
- OpenStax Calculus 2, 3.4 Partial Fractions (https://openstax.org/books/calculus-volume-2/pages/3-4-partial-fractions): Rational functions.
- OpenStax Calculus 2, 3.6 Numerical Integration (https://openstax.org/books/calculus-volume-2/pages/3-6-numerical-integration): Trapezoid and Simpson rules with error bounds.

☐ **I finished this unit**

Applications: Volume, Arc Length, Work & Center of Mass

Slice, add and take the limit: volumes of solids, lengths of curves, work against forces, fluid pressure and centers of mass.

Learning objectives

By the end of this unit you can:

- Find volumes by slicing, disks, washers and shells.
- Compute arc length.
- Compute work done by variable forces, including pumping liquids, and hydrostatic force.
- Find the center of mass of rods and thin plates.

Key concepts

- Every application has the same pattern: approximate one thin slice, write its contribution, integrate.
- Disk method: V = ∫π[f(x)]² dx about the x-axis. Shell method: V = ∫2πx f(x) dx about the y-axis.
- Arc length adds tiny hypotenuses: L = ∫√(1 + [f'(x)]²) dx.
- Work = ∫F(x) dx. Pumping: each layer of liquid is lifted its own distance.
- Pressure grows with depth, so the force on a dam is an integral over depth.
- Center of mass = moment ÷ mass.

Key equations

$$V = \int_a^b \pi [f(x)]^2 \, dx$$

V=_a^bπ[f(x)]^2 dx

Disk method

$$V = \int_a^b 2\pi x \, f(x) \, dx$$

V=_a^b2π x f(x) dx

Shell method

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx$$

L=_a^b1+[f'(x)]^2 dx

Arc length

$$W = \int_a^b F(x)\,dx$$

`W=_a^bF(x) dx`

Work by a variable force

$$\bar{x} = \frac{\int x\,\rho(x)\,dx}{\int \rho(x)\,dx}$$

`x=( x ρ(x) dx)/(ρ(x) dx)`

Center of mass of a rod

Worked examples, step by step

Example 2.1 Find the volume when the region under y = √x, 0 ≤ x ≤ 4, is spun about the x-axis.

1. Disks of radius √x: V = $\int_0^4 \pi\, x\, dx$.
2. $\pi[x^2/2]_0^4 = 8\pi$.

Answer: 8π ≈ 25.1 cubic units.

Example 2.2 A cylindrical tank of radius 1 m and height 2 m is full of water. How much work does it take to pump all the water out over the top?

1. A layer at height y has volume $\pi(1)^2\,dy$ and weight 1,000 × 9.81 × π dy newtons, and is lifted 2 − y meters.
2. $W = \int_0^2 9{,}810\pi(2 - y)\,dy = 9{,}810\pi \times 2$.
3. = 19,620π J.

Answer: ≈ 61,600 J (61.6 kJ).

Example 2.3 A rod on [0, 2] m has density ρ(x) = 1 + x kg/m. Find its center of mass.

1. Mass: $\int_0^2 (1 + x)\,dx = 2 + 2 = 4$ kg.
2. Moment: $\int_0^2 x(1 + x)\,dx = 2 + 8/3 = 14/3$.
3. $\bar{x} = (14/3)/4 = 7/6$.

Answer: $\bar{x} = 7/6 \approx 1.17$ m, toward the heavier end.

Practice problems

Work each on paper first, then check.

1. Use shells to find the volume when the region under $y = x^2$, $0 \le x \le 1$, is spun about the y-axis.

   $V = \int_0^1 2\pi x \cdot x^2 \, dx = 2\pi/4 = \pi/2$.

2. Find the arc length of y = (2/3)x^(3/2) from x = 0 to 3.

   $y' = \sqrt{x}$, so $L = \int_0^3 \sqrt{(1 + x)} \, dx = (2/3)[(1 + x)^{(3/2)}]_0^3 = (2/3)(8 - 1) = 14/3$.

3. A spring with k = 200 N/m is stretched from rest length by 0.10 m. How much work is done?

   $W = \int_0^{0.1} 200x \, dx = 100(0.01) = 1.0$ J.

4. A vertical rectangular gate 10 m wide holds back water 5 m deep. Find the total water force on it.

   $F = \int_0^5 \rho g \, y \, (10) \, dy = 1{,}000 \times 9.81 \times 10 \times 25/2 = 1.23 \times 10^6$ N.

Common misconceptions

Myth: Volume of revolution = π × area.
Actually: You must square the radius inside the integral: $\int \pi f^2 \, dx$, not $\pi(\int f \, dx)$.

Myth: Pumping work is weight × tank height.
Actually: Each layer moves a different distance, so you must integrate.

Myth: The center of mass is always the midpoint.
Actually: Only for uniform density and a symmetric shape.

Flashcards

Disk method

$V = \int \pi \, f(x)^2 \, dx$.

Washer method

$V = \int \pi (R^2 - r^2) \, dx$.

Shell method

$V = \int 2\pi x \, f(x) \, dx$.

Arc length

$L = \int \sqrt{(1 + f'^2)} \, dx$.

Work

$W = \int F\,dx$.

Center of mass

Moment divided by mass.

Self-check quiz

1. Shells about the y-axis use the integrand…

$\pi f(x)^2$

$2\pi x\, f(x)$

$f(x)^2$

$2\pi\, f(x)$

$2\pi x\, f(x)$. Circumference $2\pi x$ × height $f(x)$ × thickness dx.

2. Arc length uses…

$\int f(x)\,dx$

$\int \sqrt{1 + f'^2}\,dx$

$\int f'\,dx$

$\int \pi f^2\,dx$

$\int \sqrt{1 + f'^2}\,dx$. Pythagoras on each tiny piece.

3. Hydrostatic pressure on a dam grows…

with the square of depth

linearly with depth

not at all

with the dam’s width

linearly with depth. $P = \rho g h$; the total force then grows with depth squared.

4. The work to stretch a spring by x from rest is...

kx

$\frac{1}{2}kx^2$

kx^2

$2kx$

$\frac{1}{2}kx^2$. $\int_0^x kt\,dt$.

Try this quiz again

How this helps save the planet

Water engineers use these integrals directly: the volume of a reservoir from its shape, the energy to pump water to where it is needed, and the force a dam or flood barrier must hold back. Pumping is a significant energy cost in many water systems, so getting the integral right saves both energy and money.

Satellites now measure the height of rivers, lakes and reservoirs from orbit. NASA and CNES's SWOT mission surveys surface water so that changes in stored water can be estimated, which is slicing-and-adding on a planetary scale.

On Saving Planets: SWOT · Room for the River

Sources: OpenStax Calculus 2, 2.5 Physical Applications (https://openstax.org/books/calculus-volume-2/pages/2-5-physical-applications) · NASA: SWOT mission (https://swot.jpl.nasa.gov/)

Free learning resources

- OpenStax Calculus 2, 2.2 Determining Volumes by Slicing (https://openstax.org/books/calculus-volume-2/pages/2-2-determining-volumes-by-slicing): Disks and washers.
- OpenStax Calculus 2, 2.3 Volumes of Revolution: Cylindrical Shells (https://openstax.org/books/calculus-volume-2/pages/2-3-volumes-of-revolution-cylindrical-shells): Shells.
- OpenStax Calculus 2, 2.4 Arc Length of a Curve and Surface Area (https://openstax.org/books/calculus-volume-2/pages/2-4-arc-length-of-a-curve-and-surface-area): Arc length.
- OpenStax Calculus 2, 2.5 Physical Applications (https://openstax.org/books/calculus-volume-2/pages/2-5-physical-applications): Work, pumping and fluid force.
- OpenStax Calculus 2, 2.6 Moments and Centers of Mass (https://openstax.org/books/calculus-volume-2/pages/2-6-moments-and-centers-of-mass): Centroids.

Watch the lecture

Clip 1: Introduction to Arc Length (https://ocw.mit.edu/courses/18-01sc-single-variable-calculus-fall-2010/resources/clip-1-introduction-to-arc-length/) (in English)

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- [ ] **I finished this unit**

Improper Integrals

Integrals over infinite intervals or across infinite spikes: when they add up to a finite number, and when they do not.

Learning objectives

By the end of this unit you can:

- Evaluate improper integrals as limits.
- Decide convergence with the p-test and comparison.
- Apply improper integrals to escape energy and to average lifetimes.

Key concepts

- ∫a^∞ f dx means the limit of ∫aᵇ f dx as b → ∞. If the limit is finite, the integral converges.
- ∫₁^∞ dx/xᵖ converges exactly when p > 1. ∫₀¹ dx/xᵖ converges exactly when p < 1.
- Comparison: if 0 ≤ f ≤ g and ∫g converges, so does ∫f.
- An infinite region can have finite area.

Key equations

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^{b} f(x)\,dx$$

`_a^∞f(x) dx=_b∞_a^bf(x) dx`

Improper integral

$$\int_1^{\infty} \frac{dx}{x^p} = \frac{1}{p-1}\ (p>1)$$

`_1^∞(dx)/(x^p)=1p-1\ (p>1)`

p-integral

$$W_{\infty} = \int_R^{\infty} \frac{GMm}{r^2}\,dr = \frac{GMm}{R}$$

`W_∞=_R^∞(GMm)/(r^2) dr=GMm/R`

Work to escape a planet’s gravity

Worked examples, step by step

Example 3.1 Evaluate $\int_1^\infty dx/x^2$.

1. $\int_1^b x^{-2}\,dx = 1 - 1/b$.
2. As $b \to \infty$, $1/b \to 0$.

Answer: 1 (converges).

Example 3.2 Does $\int_1^\infty dx/x$ converge?

1. $\int_1^b dx/x = \ln b$.
2. $\ln b \to \infty$.

Answer: No, it diverges, even though $1/x \to 0$.

Example 3.3 How much energy per kilogram does it take to escape Earth's gravity from its surface (ignoring air and rotation)?

1. $W/m = \int_R^\infty GM/r^2\,dr = GM/R$.
2. $GM = 3.986 \times 10^{14}\ m^3/s^2$, $R = 6.371 \times 10^6$ m.
3. $W/m = 6.26 \times 10^7$ J/kg.

Answer: About 62.6 MJ per kilogram, finite even though the distance is infinite. Setting $\tfrac{1}{2}v^2$ equal to it gives the 11.2 km/s escape speed.

Practice problems

Work each on paper first, then check.

1. $\int_0^\infty e^{-x}\,dx$

   $[-e^{-x}]_0^\infty = 0 + 1 = 1$.

2. $\int_0^1 dx/\sqrt{x}$

   $[2\sqrt{x}]_0^1 = 2$ (converges; $p = \tfrac{1}{2} < 1$).

3. $\int_2^\infty dx/x^3$

   $[-1/(2x^2)]_2^\infty = 1/8$.

4. Does $\int_1^\infty dx/(x^2 + 1)$ converge?

   Yes: it is less than $1/x^2$, which converges. In fact it equals $\pi/4$.

5. For exponential decay with rate λ, the mean lifetime is $\int_0^\infty t\,\lambda e^{-\lambda t}\,dt$. Evaluate it.

   By parts: $1/\lambda$.

Common misconceptions

Myth: If $f(x) \to 0$ as $x \to \infty$, the integral converges.

Actually: $1/x \to 0$ but $\int_1^\infty dx/x$ diverges.

Myth: An infinitely long region has infinite area.

Actually: $\int_1^\infty dx/x^2 = 1$.

Myth: $\int_{-1}^{1} dx/x^2 = -2$ by the usual formula.

Actually: The integrand blows up at 0, so the integral is improper, and it diverges.

Flashcards

Improper integral

Infinite interval or infinite integrand; defined by a limit.

Converges

The limit exists and is finite.

p-test (∞)

$\int_1^\infty x^{-p}$ converges iff $p > 1$.

p-test (0)

$\int_0^1 x^{-p}$ converges iff $p < 1$.

Comparison test

Smaller than convergent ⇒ converges.

Escape energy

GM/R per kg.

Self-check quiz

1. $\int_1^\infty dx/x^{1.5}$...

diverges

converges to 2

converges to 1

converges to 1.5

converges to 2. 1/(p − 1) = 1/0.5 = 2.

2. $\int_1^\infty dx/x$...

converges to 1

diverges

converges to 0

equals ln 2

diverges. ln b → ∞.

3. $\int_0^\infty e^{(-2x)}\,dx$ =

2

1/2

1

∞

1/2. $[-e^{(-2x)/2}]_0^\infty$ = 1/2.

4. An improper integral is defined as...

a sum

a limit of proper integrals

a derivative

always infinite

a limit of proper integrals. Take the limit of $\int_a^b$.

Try this quiz again

How this helps save the planet

Improper integrals give the energy to leave a planet, the number behind every launch of a weather or climate satellite. It is finite, which is why spaceflight is possible at all.

Mean lifetimes, also improper integrals, describe how long substances stick around: how long a pollutant persists in a lake, or how long a radioactive tracer used to date ice and ocean water remains.

On Saving Planets: Physics I, Unit 7: orbits · Delta-v calculator

Sources: OpenStax Calculus 2, 3.7 Improper Integrals (https://openstax.org/books/calculus-volume-2/pages/3-7-improper-integrals) · NASA Earth fact sheet (https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html)

Free learning resources

- OpenStax Calculus 2, 3.7 Improper Integrals (https://openstax.org/books/calculus-volume-2/pages/3-7-improper-integrals): Free textbook section with comparison tests.

Watch the lecture

Lecture 36: Improper Integrals (https://ocw.mit.edu/courses/18-01-single-variable-calculus-fall-2006/resources/lecture-36-improper-integrals/) (in English)

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☐ **I finished this unit**

Introduction to Differential Equations

Equations for rates of change: exponential growth and decay, cooling, mixing, and the logistic model of populations.

Learning objectives

By the end of this unit you can:

- Verify solutions and solve initial-value problems.
- Read direction fields and use Euler’s method.
- Solve separable equations, including mixing and cooling problems.
- Use the logistic equation and interpret its carrying capacity.

Key concepts

- A differential equation links a function to its derivatives. Its solution is a function, usually a family fixed by an initial condition.
- Separable equations: put all y on one side and all x on the other, then integrate.
- y' = ky gives $y = y_0 e^{(kt)}$: growth if k > 0, decay if k < 0.
- Mixing: rate of change of a pollutant = rate in − rate out.
- Logistic: P' = rP(1 − P/K). Growth slows as P nears the carrying capacity K; it is fastest at P = K/2.
- Euler’s method steps along the slope: $y_{n+1} = y_n + h\, f(t_n, y_n)$. Smaller steps, smaller error.

Key equations

$$\frac{dy}{dt} = ky \;\Rightarrow\; y = y_0 e^{kt}$$

`dy/dt=ky\ \ y=y_0e^kt`

Exponential growth and decay

$$\frac{dT}{dt} = -k(T - T_s)$$

`dT/dt=-k(T-T_s)`

Newton’s law of cooling

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \quad P(t) = \frac{K}{1 + \frac{K - P_0}{P_0} e^{-rt}}$$

`dP/dt=rP(1- PK), P(t)=1+(K-P_0)/(P_0)e^-rt`

Logistic equation and its solution

$$y_{n+1} = y_n + h\, f(t_n, y_n)$$

`y_n+1=y_n+h f(t_n,y_n)`

Euler’s method

Worked examples, step by step

Example 4.1 A lake section holds 1,000 m³ of water with 50 kg of a dissolved pollutant. Clean water flows in at 10 m³/day and well-mixed water flows out at the same rate. How long until half the pollutant is gone?

1. dS/dt = −(10/1,000)S = −S/100.
2. S(t) = 50e^(−t/100).
3. Half: e^(−t/100) = ½, so t = 100 ln 2.

Answer: ≈ 69.3 days (a simplified, well-mixed model).

Example 4.2 A fish stock follows P' = 0.5P(1 − P/1,000) with P(0) = 100. Find P(4).

1. P(t) = 1,000/(1 + 9e^(−0.5t)).
2. P(4) = 1,000/(1 + 9e^(−2)) = 1,000/2.218.

Answer: ≈ 451 fish, near the fastest-growth level K/2 = 500.

Example 4.3 Use Euler’s method with h = 0.5 to estimate y(1) for y' = y, y(0) = 1.

1. $y_1 = 1 + 0.5(1) = 1.5$.
2. $y_2 = 1.5 + 0.5(1.5) = 2.25$.

Answer: 2.25, versus the exact e ≈ 2.718; halving h shrinks the error.

Practice problems

Work each on paper first, then check.

1. Coffee at 90 °C cools in a 20 °C room with k = 0.1 per minute. Find its temperature after 10 minutes.

   T = 20 + 70e^(−1) ≈ 45.8 °C.

2. Solve y' = xy with y(0) = 2, and find y(1).

dy/y = x dx, so y = 2e^(x²/2); y(1) = 2e^0.5 ≈ 3.30.

3. Solve y' = −2y, y(0) = 5. Find y(1).

   y = 5e^(−2t); y(1) ≈ 0.677.

4. For P' = rP(1 − P/K), at what population is growth fastest?

   rP(1 − P/K) is a downward parabola in P with its peak at P = K/2.

Common misconceptions

Myth: The solution of a differential equation is a number.

Actually: It is a function (or family of functions).

Myth: Euler's method gives the exact answer if you are careful.

Actually: It has a step-size error; it converges only as h → 0.

Myth: Logistic populations stop growing at K/2.

Actually: Growth is fastest at K/2 and slows to zero only as P approaches K.

Flashcards

Differential equation

An equation involving a function and its derivatives.

Initial-value problem

A DE plus a starting value.

Separable

g(y) dy = h(x) dx.

Logistic equation

P' = rP(1 − P/K).

Carrying capacity

K, the long-run limit of a logistic population.

Euler's method

$y_{n+1} = y_n + h\, f(t_n, y_n)$.

Self-check quiz

1. The general solution of y' = 3y is…

$3e^x$

Ce^(3x)

e^(3x) + C

3x + C

Ce^(3x). Separate and integrate.

2. In the logistic model, K is the…

growth rate

initial population

carrying capacity

half-life

carrying capacity. The level the population approaches.

3. One Euler step from (0, 2) with y' = y and h = 0.1 gives…

2.1

2.2

2.02

2.0

2.2. 2 + 0.1 × 2.

4. dy/dx = x^2y is…

not separable

separable

linear with constant coefficients only

logistic

separable. $dy/y = x^2\,dx$.

Try this quiz again

How this helps save the planet

Population models built on the logistic equation are part of how fishery scientists assess stocks and set catch limits that let populations stay healthy, which protects ocean ecosystems and the people who depend on them.

Mixing equations like Example 4.1 are the first step in estimating how quickly a lake or estuary flushes out a pollutant once the source is stopped.

Weather and climate models solve very large systems of differential equations with step-by-step numerical methods, descendants of Euler's method.

On Saving Planets: Lakes hub · Oceans hub

Sources: OpenStax Calculus 2, 4.4 The Logistic Equation (https://openstax.org/books/calculus-volume-2/pages/4-4-the-logistic-equation) · NOAA Fisheries: population assessments (https://www.fisheries.noaa.gov/topic/population-assessments)

Free learning resources

- OpenStax Calculus 2, 4.1 Basics of Differential Equations (https://openstax.org/books/calculus-volume-2/pages/4-1-basics-of-differential-equations): Free textbook section.
- OpenStax Calculus 2, 4.2 Direction Fields and Numerical Methods (https://openstax.org/books/calculus-volume-2/pages/4-2-direction-fields-and-numerical-methods): Euler's method.
- OpenStax Calculus 2, 4.3 Separable Equations (https://openstax.org/books/calculus-volume-2/pages/4-3-separable-equations): Mixing and cooling.
- OpenStax Calculus 2, 4.4 The Logistic Equation (https://openstax.org/books/calculus-volume-2/pages/4-4-the-logistic-equation): Population models.

Watch the lecture

Lecture 16: Differential Equations (https://ocw.mit.edu/courses/18-01-single-variable-calculus-fall-2006/resources/lecture-16-differential-equations/) (in English)

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☐ I finished this unit

Sequences & Series

When does adding infinitely many numbers give a finite total? Geometric series, the harmonic series and the convergence tests.

Learning objectives

By the end of this unit you can:

- Find limits of sequences.
- Sum geometric and telescoping series.
- Apply the divergence, integral, comparison, alternating-series, ratio and root tests.
- Distinguish absolute from conditional convergence.

Key concepts

- A series Σa_n converges if its partial sums approach a limit.
- Geometric: $\Sigma\, ar^n$ $(n \ge 0) = a/(1 - r)$ when $|r| < 1$.
- Divergence test: if a_n does not $\to 0$, the series diverges. The converse is false: the harmonic series $\Sigma 1/n$ diverges.
- p-series $\Sigma 1/n^p$ converges iff $p > 1$ (by the integral test).
- Ratio test: if $|a_{n+1}/a_n| \to L < 1$ the series converges absolutely; $L > 1$ diverges; $L = 1$ says nothing.
- Alternating series with terms decreasing to 0 converge; $\Sigma(-1)^{n+1}/n = \ln 2$.

Key equations

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}\ (|r| < 1)$$

`_n=0^∞ar^n=(a)/(1-r)\ (|r|<1)`

Geometric series

$$\sum \frac{1}{n^p} \text{ converges } \iff p > 1$$

p-series

$$L = \lim \left| \frac{a_{n+1}}{a_n} \right|$$

`L=|_n+1a_n|`

Ratio test

Worked examples, step by step

Example 5.1 Sum $\Sigma\ (1/2)^n$ from $n = 0$.

1. Geometric with $a = 1$, $r = ½$.
2. $1/(1 - ½) = 2$.

Answer: 2.

Example 5.2 Each year a lake loses 10% of a stored pollutant, and 5 tonnes are added. What level does it settle at?

1. After many years the amount is $5 + 5(0.9) + 5(0.9)^2 + \ldots$
2. Geometric: $5/(1 - 0.9)$.

Answer: 50 tonnes. Cutting the yearly input in half halves the long-run level (a simplified model).

Example 5.3 Does $\Sigma\ n/2^n$ converge?

1. Ratio: $[(n + 1)/2^{n+1}]/[n/2^n] = (n + 1)/(2n) \to ½$.
2. $½ < 1$.

Answer: Converges (its sum is 2).

Practice problems

Work each on paper first, then check.

1. $\Sigma\ 0.9(0.1)^n$ from $n = 0$

   $0.9/(1 - 0.1) = 1$. So $0.999\ldots = 1$.

2. Does $\Sigma\ 1/n^2$ converge?

   Yes, $p = 2 > 1$ (the sum is $\pi^2/6$).

3. Does $\Sigma\ n/(n + 1)$ converge?

   No: the terms $\to 1$, not 0 (divergence test).

4. Sum $\Sigma\ 3(2/3)^n$ from $n = 0$.

   $3/(1 - 2/3) = 9$.

5. Does $\Sigma (-1)^{n+1}/n$ converge absolutely?

   No. It converges (to ln 2), but $\Sigma 1/n$ diverges, so the convergence is conditional.

Common misconceptions

Myth: If the terms go to 0, the series converges.

Actually: The harmonic series is the counterexample.

Myth: A ratio-test limit of 1 means divergence.

Actually: $L = 1$ is inconclusive; use another test.

Myth: Rearranging terms never changes a sum.

Actually: For conditionally convergent series it can.

Flashcards

Sequence

An ordered list $a_1, a_2, \ldots$

Series

Sum of a sequence; limit of partial sums.

Geometric series

$a/(1 - r)$ for $|r| < 1$.

Harmonic series

$\Sigma 1/n$, diverges.

Ratio test

$L < 1$ converges, $L > 1$ diverges, $L = 1$ inconclusive.

Conditional convergence

Converges, but not absolutely.

Self-check quiz

1. $\Sigma (1/3)^n$ from n = 0 =

1/3

3/2

3

2

3/2. 1/(1 − 1/3).

2. The harmonic series Σ1/n...

converges to 1

diverges

converges to ln 2

converges to e

diverges. p = 1, integral test.

3. Σ 1/n^1.01...

converges

diverges

cannot be decided

equals 1

converges. p > 1.

4. The ratio test for Σ 1/n! gives L =

1

0

∞

1/2

0. 1/(n + 1) → 0, so it converges.

Try this quiz again

How this helps save the planet

Geometric series describe anything that is partly carried over from one period to the next, as in Example 5.2. That simple model shows a key fact of pollution and of greenhouse gases: the long-run level depends on the input rate, so cutting inputs lowers where things settle.

Real systems are more complex than one geometric series, and scientists use full models, but the series is the right first intuition.

On Saving Planets: Lake Erie algal bloom forecast · Lakes hub

Sources: OpenStax Calculus 2, 5.2 Infinite Series (https://openstax.org/books/calculus-volume-2/pages/5-2-infinite-series) · NOAA GML: trends in atmospheric CO_2 (https://gml.noaa.gov/ccgg/trends/)

Free learning resources

- OpenStax Calculus 2, 5.1 Sequences (https://openstax.org/books/calculus-volume-2/pages/5-1-sequences): Free textbook section.
- OpenStax Calculus 2, 5.2 Infinite Series (https://openstax.org/books/calculus-volume-2/pages/5-2-infinite-series): Geometric and telescoping series.
- OpenStax Calculus 2, 5.6 Ratio and Root Tests (https://openstax.org/books/calculus-volume-2/pages/5-6-ratio-and-root-tests): Ratio and root tests.

- [ ] **I finished this unit**

Power Series & Taylor Series

Turn functions into infinite polynomials. Taylor series are how calculators compute, and how physicists make hard problems simple.

Learning objectives

By the end of this unit you can:

- Find the radius and interval of convergence of a power series.
- Build Taylor and Maclaurin series and polynomials.
- Bound the error of a Taylor approximation.
- Use series to approximate integrals and to justify small-angle approximations.

Key concepts

- A power series $\Sigma c_n(x - a)^n$ converges inside a radius R and diverges outside; check the endpoints separately.
- Taylor series: $c_n = f^{(n)}(a)/n!$. Around a = 0 it is called a Maclaurin series.
- Key series: $e^x = \Sigma x^n/n!$, $\sin x = x - x^3/3! + x^5/5! - \ldots$, $\cos x = 1 - x^2/2! + x^4/4! - \ldots$, $\ln(1 + x) = x - x^2/2 + x^3/3 - \ldots$ ($|x| < 1$).
- Taylor's remainder: the error is at most $M|x - a|^{n+1}/(n + 1)!$, where M bounds the next derivative.
- Keeping only the first term turns $\sin \theta$ into θ, the small-angle approximation behind the simple pendulum formula.

Key equations

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$

`f(x)=_n=0^∞^(n)(a)n!(x-a)^n`

Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

`e^x=_n=0^∞(x^n)/(n!)`

Exponential series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

`x=x-(x^3)/(3!)+(x^5)/(5!)-·s`

Sine series

$$|R_n(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!}$$

Taylor remainder bound

Worked examples, step by step

Example 6.1 Estimate e^0.1 with the first four terms of its series.

1. 1 + 0.1 + 0.01/2 + 0.001/6.
2. = 1.105167.

Answer: 1.105167, versus the true 1.105171.

Example 6.2 Estimate sin 0.5 with three terms and bound the error.

1. 0.5 − 0.125/6 + 0.03125/120 = 0.479427.
2. Next term bound: $0.5^7/7! = 1.55 \times 10^{-6}$.

Answer: 0.479427, error under 1.6×10^{-6} (true value 0.479426).

Example 6.3 Find the radius of convergence of $\Sigma\, x^n/3^n$.

1. Ratio: $|x|/3 < 1$.
2. Endpoints $x = \pm 3$ give terms ± 1, which do not $\to 0$.

Answer: $R = 3$; it converges for $-3 < x < 3$.

Practice problems

Work each on paper first, then check.

1. Write the Maclaurin polynomial of degree 4 for cos x and use it for cos 0.2.

   $1 - x^2/2 + x^4/24$; $\cos 0.2 \approx 1 - 0.02 + 0.0000667 = 0.980067$ (true 0.980067).

2. Find the Maclaurin series of $1/(1 - x)$.

   Σx^n for $|x| < 1$ (geometric).

3. Find the radius of convergence of $\Sigma\, x^n/n$.

   $R = 1$ (ratio $n/(n+1) \to 1$).

4. Use a series to estimate $\int_0^1 e^{\wedge}(-x^2)\, dx$ with three terms.

   $\int(1 - x^2 + x^4/2)\, dx = 1 - 1/3 + 1/10 = 0.7667$. More terms approach 0.7468.

Common misconceptions

Myth: A Taylor polynomial is accurate everywhere.

Actually: It is best near the center a and can be terrible far away or outside the radius of convergence.

Myth: Every smooth function equals its Taylor series.

Actually: Not always, though it holds for e^x, sin x and cos x everywhere.

Myth: The radius of convergence decides the endpoints too.

Actually: Endpoints must be checked separately.

Flashcards

Power series

$\Sigma c_n(x - a)^n$.

Radius of convergence

R such that the series converges for $|x - a| < R$.

Taylor coefficient

$f^{(n)}(a)/n!$.

Maclaurin series

Taylor series at a = 0.

Small-angle approximation

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \theta^2/2$.

Remainder bound

$M|x - a|^{n+1}/(n+1)!$.

Self-check quiz

1. The Maclaurin series of e^x is...

Σx^n

$\Sigma x^n/n!$

$\Sigma(-1)^n x^n$

$\Sigma x^n/n$

$\Sigma x^n/n!$. Every derivative of e^x at 0 is 1.

2. For small θ, $\sin\theta \approx$...

1

θ

θ^2

$1 - \theta$

θ. The first term of the series.

3. The coefficient of x^3 in the series for sin x is...

1/3

−1/6

1/6

−1/3

−1/6. −1/3!.

4. Σx^n converges for...

all x

$|x| < 1$

$|x| \leq 1$

$x > 0$

$|x| < 1$. Geometric series.

Try this quiz again

How this helps save the planet

Taylor series are how physicists turn hard equations into usable ones. The pendulum period $T = 2\pi\sqrt{(L/g)}$ in Physics I relies on $\sin\theta \approx \theta$, and the same kind of linearization underlies much of how engineers analyze vibrations in wind turbines, bridges and aircraft.

Satellite orbit and climate models expand complicated effects into series and keep the terms that matter. Knowing the error bound tells you when an approximation is good enough to trust.

On Saving Planets: Physics I, Unit 8: oscillations · Pendulum period calculator

Sources: OpenStax Calculus 2, 6.3 Taylor and Maclaurin Series (https://openstax.org/books/calculus-volume-2/pages/6-3-taylor-and-maclaurin-series) · OpenStax University Physics 1, 15.4 Pendulums (https://openstax.org/books/university-physics-volume-1/pages/15-4-pendulums)

Free learning resources

- OpenStax Calculus 2, 6.3 Taylor and Maclaurin Series (https://openstax.org/books/calculus-volume-2/pages/6-3-taylor-and-maclaurin-series): Free textbook section.
- MIT OpenCourseWare 18.01SC (https://ocw.mit.edu/courses/18-01sc-single-variable-calculus-fall-2010/): Series lectures and problem sets.

Watch the lecture

Lecture 38: Taylor's Series (https://ocw.mit.edu/courses/18-01-single-variable-calculus-fall-2006/resources/lecture-38-taylors-series/) (in English)

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☐ I finished this unit

Parametric Equations & Polar Coordinates

Describe paths by time and curves by angle, including the ellipse every satellite and planet follows.

Learning objectives

By the end of this unit you can:

- Graph parametric curves, find dy/dx and arc length.
- Convert between polar and Cartesian coordinates.
- Find areas and arc lengths in polar coordinates.
- Describe a Kepler orbit in polar form.

Key concepts

- A parametric curve gives x(t) and y(t); dy/dx = (dy/dt)/(dx/dt).
- Arc length: ∫√(x'² + y'²) dt.
- Polar: x = r cos θ, y = r sin θ. Area of a polar region: ½∫r² dθ.
- An orbit around one focus is r = p/(1 + e cos θ), where e is the eccentricity.
- Kepler’s second law (equal areas in equal times) is the statement that ½r² dθ/dt is constant.

Key equations

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

`dy/dx=(dy/dt)/(dx/dt)`

Slope of a parametric curve

$$L = \int_a^b \sqrt{(x')^2 + (y')^2}\, dt$$

`L=_a^b(x')^2+(y')^2 dt`

Parametric arc length

$$A = \tfrac{1}{2}\int_\alpha^\beta r^2\, d\theta$$

`A=12_α^β r^2 dθ`

Polar area

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

`r=(a(1-e^2))/(1+eθ)`

Conic orbit in polar form

Worked examples, step by step

Example 7.1 Find the slope of x = t², y = t³ at t = 1.

1. dx/dt = 2t, dy/dt = 3t².
2. dy/dx = 3t²/(2t) = 3t/2 = 3/2 at t = 1.

Answer: 3/2.

Example 7.2 Find the area inside the cardioid r = 1 + cos θ.

1. A = ½∫₀^(2π) (1 + cos θ)² dθ.
2. = ½∫(1 + 2cos θ + cos²θ) dθ = ½(2π + 0 + π).

Answer: 3π/2 ≈ 4.71.

Example 7.3 Earth’s orbit has a = 149.6 million km and e = 0.0167. Find its closest and farthest distances from the Sun.

1. At θ = 0: r = a(1 − e) = 149.6 × 0.9833.
2. At θ = π: r = a(1 + e) = 149.6 × 1.0167.

Answer: About 147.1 and 152.1 million km, matching NASA’s published perihelion and aphelion.

Practice problems

Work each on paper first, then check.

1. Find the length of x = cos t, y = sin t for 0 ≤ t ≤ 2π.

   ∫√(sin²t + cos²t) dt = 2π (a circle’s circumference).

2. Find the area of the circle r = 3.

   ½∫₀^(2π) 9 dθ = 9π.

3. Convert $r = 2\cos\theta$ to Cartesian form.

   $r^2 = 2r\cos\theta$ gives $x^2 + y^2 = 2x$: a circle of radius 1 centered at (1, 0), area π.

4. Eliminate t from $x = 2\cos t$, $y = \sin t$.

   $x^2/4 + y^2 = 1$, an ellipse.

Common misconceptions

Myth: $dy/dx = dy/dt \times dx/dt$.
Actually: Divide: $(dy/dt)/(dx/dt)$.

Myth: Polar area is $\int r\, d\theta$.
Actually: It is $\frac{1}{2}\int r^2\, d\theta$, from thin pie slices.

Myth: The Sun is at the center of Earth's orbit.
Actually: It is at one focus of the ellipse.

Flashcards

Parametric curve

$x = x(t)$, $y = y(t)$.

Parametric slope

$(dy/dt)/(dx/dt)$.

Polar coordinates

(r, θ); $x = r\cos\theta$, $y = r\sin\theta$.

Polar area

$\frac{1}{2}\int r^2\, d\theta$.

Eccentricity

$e = 0$ circle, $0 < e < 1$ ellipse.

Kepler's second law

Equal areas in equal times.

Self-check quiz

1. Polar area uses...

$\int r\, d\theta$

$\frac{1}{2}\int r^2\, d\theta$

$\int r^2\, d\theta$

$2\pi\int r\, d\theta$

$\frac{1}{2}\int r^2\, d\theta$. Each slice is approximately a sector of area $\frac{1}{2}r^2\, d\theta$.

2. $x = r\cos\theta$, $y = \ldots$

$r\tan\theta$

$r\sin\theta$

$\theta\sin r$

r/θ

$r\sin\theta$. Polar to Cartesian.

3. For an orbit $r = p/(1 + e\cos\theta)$, $e = 0$ gives...

a parabola

a circle

a line

a hyperbola

a circle. $r = p$ is constant.

4. Kepler's second law says $r^2\, d\theta/dt$ is...

zero

constant

increasing

equal to e

constant. Equal areas in equal times.

Try this quiz again

How this helps save the planet

Every Earth-observing satellite follows a path described by the polar orbit equation in this unit. Mission designers choose the orbit's shape and tilt so instruments revisit the same places under similar lighting, which makes changes on the ground easier to detect.

Parametric equations describe the ground track a satellite traces over Earth, which decides which forests, coasts and ice sheets it can see and when.

On Saving Planets: Orbits guide · Landsat 9

Sources: NASA Earth fact sheet (https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html) · OpenStax Calculus 2, 7.4 Area and Arc Length in Polar Coordinates (https://openstax.org/books/calculus-volume-2/pages/7-4-area-and-arc-length-in-polar-coordinates)

Free learning resources

- OpenStax Calculus 2, 7.1 Parametric Equations (https://openstax.org/books/calculus-volume-2/pages/7-1-parametric-equations): Free textbook section.
- OpenStax Calculus 2, 7.3 Polar Coordinates (https://openstax.org/books/calculus-volume-2/pages/7-3-polar-coordinates): Polar graphs.
- OpenStax Calculus 2, 7.4 Area and Arc Length in Polar Coordinates (https://openstax.org/books/calculus-volume-2/pages/7-4-area-and-arc-length-in-polar-coordinates): Polar area and length.
- PhET: Kepler's Laws (https://phet.colorado.edu/en/simulations/keplers-laws): Orbits, areas and periods.

Watch the lecture

Lecture 31: Parametric Equations (https://ocw.mit.edu/courses/18-01-single-variable-calculus-fall-2006/resources/lecture-31-parametric-equations/) (in English)

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☐ **I finished this unit**

Problem Sets

These practice problems live inside each unit. Work on paper first, then open the answer.

1. Techniques of Integration
   5 problems
2. Applications: Volume, Arc Length, Work & Center of Mass
   4 problems
3. Improper Integrals
   5 problems

4. Introduction to Differential Equations
   4 problems
5. Sequences & Series
   5 problems
6. Power Series & Taylor Series
   4 problems
7. Parametric Equations & Polar Coordinates
   4 problems

Computer labs

Four short computer labs apply the units with a spreadsheet: Euler's method for a population model, Simpson's rule for a reservoir volume, Taylor-series accuracy, and Kepler's second law in polar coordinates.

These are the numerical methods scientists and engineers use daily. Sample data is labeled as such; each lab links to its source.

Safety reference: OpenStax Calculus Volume 2 (free textbook) (https://openstax.org/books/calculus-volume-2/pages/1-introduction)

Lab A: Euler’s method for a recovering fish stock

WEEK 8

Goal: Solve the logistic equation numerically and compare with the exact solution.

$$P_{n+1} = P_n + h\,r P_n\left(1 - \frac{P_n}{K}\right)$$

```
P_n+1=P_n+h rP_n(1-(P_n)/(K))
```

Euler’s method for the logistic equation

Procedure

1. Set r = 0.5, K = 1,000, P_0 = 100 and h = 1 in a spreadsheet.
2. Fill down Euler steps to t = 4.
3. Add a column with the exact solution and the error.
4. Repeat with h = 0.5 and h = 0.25; how does the error change?

Data table

t	Euler P (h = 1)
0	100
1	145.0
2	207.0
3	289.1
4	391.9

Computed model values (not a real fishery). Exact P(4) = 450.9.

Sample calculation

1. P_1 = 100 + 0.5 × 100 × 0.9 = 145.0.
2. P_2 = 145 + 0.5 × 145 × 0.855 = 207.0.
3. Euler underestimates here (391.9 vs 450.9) because the curve bends upward during early growth; halving h roughly halves the error.

> **Safety**
>
> - None beyond normal computer use.

Home or virtual alternative

- A spreadsheet is enough.

OpenStax Calculus 2, 4.2 Direction Fields and Numerical Methods (https://openstax.org/books/calculus-volume-2/pages/4-2-direction-fields-and-numerical-methods)

Planet link: Numerical population models support fishery management. NOAA Fisheries: population assessments (https://www.fisheries.noaa.gov/topic/population-assessments)

☐ **I completed this lab (wet or virtual)**

Lab B: Reservoir volume with Simpson’s rule

WEEK 5

Goal: Estimate a reservoir’s volume from surface-area measurements at different depths.

$$V = \int_0^H A(h)\,dh \approx \frac{\Delta h}{3}[A_0 + 4A_1 + 2A_2 + 4A_3 + A_4]$$

```
V=_0^HA(h) dh≈Δ h3[A_0+4A_1+2A_2+4A_3+A_4]
```

Volume as the integral of cross-sectional area

Procedure

1. Enter the sample areas at 0, 2, 4, 6 and 8 m above the bottom.
2. Apply Simpson’s rule with Δh = 2 m.
3. Compare with the trapezoid rule.

Data table

Height h (m)	Surface area A (km²)
0	0.0
2	1.2
4	2.6
6	4.0
8	5.6

Sample values for practice, not a real reservoir.

Sample calculation

1. Simpson: (2/3)[0 + 4(1.2) + 2(2.6) + 4(4.0) + 5.6] = (2/3)(31.6) = 21.07 km²·m.
2. 1 km²·m = 10^6 m³, so V ≈ 2.11 × 10^7 m³ (21 million cubic meters).
3. Trapezoid: 2[0/2 + 1.2 + 2.6 + 4.0 + 5.6/2] = 21.2 km²·m, close to Simpson.

> **Safety**
>
> - None beyond normal computer use.

Home or virtual alternative

- A spreadsheet is enough.

NASA: SWOT mission (https://swot.jpl.nasa.gov/)

Planet link: Surface-water satellites such as SWOT help estimate changes in water stored in lakes and reservoirs.

- [ ] **I completed this lab (wet or virtual)**

Lab C: How good is a Taylor polynomial?

WEEK 13

Goal: Compare Taylor polynomials of sin x with the true value and with the remainder bound.

$$P_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$$

```
P_5(x)=x-(x^3)/(6)+(x^5)/(120)
```

Degree-5 Maclaurin polynomial of sin x

Procedure

1. Tabulate sin x, $P_1 = x$, P_3 and P_5 for x from 0 to π in steps of 0.25.
2. Plot the errors.
3. Find where the small-angle approximation sin x ≈ x is off by 1%.

Data table

x	sin x	$P_5(x)$
0.5	0.479426	0.479427
1.0	0.841471	0.841667
2.0	0.909297	0.933333

Computed values.

Sample calculation

1. At x = 0.5, P_5 errs by about 10^{-6}; at x = 2 by 0.024. Accuracy falls fast away from 0.
2. sin x ≈ x is within 1% up to about x = 0.24 rad (14°), which is why the pendulum formula is used for small swings.

> **Safety**
>
> - None beyond normal computer use.

Home or virtual alternative

- Desmos sliders work well.

PhET: Pendulum Lab (https://phet.colorado.edu/en/simulations/pendulum-lab)

Planet link: Knowing an approximation's error is how engineers decide which simplified model to trust.

- [ ] **I completed this lab (wet or virtual)**

Lab D: Kepler’s second law in polar coordinates

WEEK 14

Goal: Show numerically that an elliptical orbit sweeps area faster near the focus in angle but at a constant rate in time.

$$A(\theta_1, \theta_2) = \tfrac{1}{2}\int_{\theta_1}^{\theta_2} r(\theta)^2\, d\theta, \quad r = \frac{a(1-e^2)}{1+e\cos\theta}$$

```
A(θ_1,θ_2)=12_θ_1^θ_2r(θ)^2 dθ, r=(a(1-e^2))/(1+eθ)
```

Swept area of an orbit

Procedure

1. Use a = 1 and e = 0.5.
2. Compute the swept area from θ = 0 to 0.5 and from θ = π − 0.25 to π + 0.25 (same 0.5 rad wide) with Simpson’s rule.
3. Explain why equal angles do not give equal areas, and so take unequal times.

Data table

Arc	Swept area
θ = 0 to 0.5 (near the Sun)	0.0643
θ = π ± 0.25 (far side)	0.5513

Computed values for a = 1, e = 0.5.

Sample calculation

1. Near the Sun r ≈ 0.5, far side r ≈ 1.5: area ∝ r^2, so the far arc sweeps about 8.6 times more area (0.5513 ÷ 0.0643).
2. Since area grows at a constant rate in time, the planet spends about 8.6 times longer crossing the far 0.5 rad: it moves slowest at aphelion.

> **Safety**
>
> - None beyond normal computer use.

Home or virtual alternative

- A spreadsheet or Desmos is enough.

PhET: Kepler’s Laws (https://phet.colorado.edu/en/simulations/keplers-laws)

Planet link: Orbit geometry decides when a satellite can see a place on Earth.

- [ ] **I completed this lab (wet or virtual)**

Exam prep practice sets

Time yourself: about 10 minutes per problem. Then compare with the full solutions.

Midterm practice set (Units 1–3)

1. Find $\int_0^1 x^2 e^x \, dx$.
    1. Parts twice: $\int x^2 e^x \, dx = (x^2 - 2x + 2)e^x$.
    2. From 0 to 1: $e - 2$.

    Answer: $e - 2 \approx 0.718$.

2. Find $\int_0^1 x/(x + 1) \, dx$.
    1. $x/(x + 1) = 1 - 1/(x + 1)$.
    2. $[x - \ln(x + 1)]_0^1 = 1 - \ln 2$.

    Answer: $1 - \ln 2 \approx 0.307$.

3. Find the volume when the region under $y = x^2$, $0 \le x \le 2$, is spun about the x-axis.
    1. Disks: $\int_0^2 \pi x^4 \, dx = \pi \times 32/5$.

    Answer: $32\pi/5 \approx 20.1$.

4. Evaluate $\int_2^\infty dx/x^3$.
    1. $[-1/(2x^2)]_2^\infty = 1/8$.

    Answer: 1/8.

Final practice set (all units)

1. Solve $y' = -2y$, $y(0) = 5$, and find $y(1)$.
    1. $y = 5e^{(-2t)}$.

    Answer: $5e^{(-2)} \approx 0.677$.

2. Sum $\Sigma\, 3(2/3)^n$ from $n = 0$.
    1. Geometric: $3/(1 - 2/3)$.

Answer: 9.

3. Estimate cos 0.2 with the degree-4 Maclaurin polynomial.

    1. 1 − 0.04/2 + 0.0016/24.

    Answer: 0.980067.

4. Find the area inside r = 3.

    1. ½∫₀^(2π) 9 dθ.

    Answer: 9π ≈ 28.3.

5. Find the interval of convergence of Σ xⁿ/3ⁿ.

    1. Ratio: |x|/3 < 1; endpoints diverge.

    Answer: −3 < x < 3.

6. A mixing tank's pollutant follows S' = −S/50 with S(0) = 20 kg. When is S = 5 kg?

    1. S = 20e^(−t/50); e^(−t/50) = 1/4.
    2. t = 50 ln 4.

    Answer: ≈ 69.3 time units.

Formula sheet

$$\int u\,dv = uv - \int v\,du$$

$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x),\ \cos^2 x = \tfrac{1}{2}(1+\cos 2x)$$

$$S_n = \tfrac{\Delta x}{3}[f_0 + 4f_1 + 2f_2 + \cdots + f_n]$$

$$V = \int \pi f^2\,dx,\ V = \int 2\pi x f\,dx$$

$$L = \int \sqrt{1+f'^2}\,dx$$

$$W = \int F\,dx$$

$$\int_1^\infty x^{-p}dx < \infty \iff p > 1$$

$$y' = ky \Rightarrow y = y_0 e^{kt}$$

$$P' = rP(1-P/K)$$

$$y_{n+1} = y_n + hf(t_n, y_n)$$

$$\sum ar^n = a/(1-r)$$

$$L = \lim |a_{n+1}/a_n|$$

$$e^x = \sum x^n/n!$$

$$\sin x = \sum (-1)^n x^{2n+1}/(2n+1)!$$

$$\cos x = \sum (-1)^n x^{2n}/(2n)!$$

$$dy/dx = (dy/dt)/(dx/dt)$$

$$A = \tfrac{1}{2}\int r^2\,d\theta$$

Constants

Constant	Value
e	2.718281828…
π	3.141592654…
ln 2	0.693147…
Earth’s orbital eccentricity	0.0167
Earth’s semi-major axis	149.6 million km
Density of water	1,000 kg/m³

Source: NASA Earth fact sheet (https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html)

Graphing and data tools

Use these free tools alongside the course.

- Desmos graphing calculator (https://www.desmos.com/calculator): Graph parametric and polar curves, sliders and Taylor polynomials.
- MIT OpenCourseWare 18.01SC (https://ocw.mit.edu/courses/18-01sc-single-variable-calculus-fall-2010/): Free MIT lectures and problem sets for integration and series.
- MIT OpenCourseWare 18.02SC (https://ocw.mit.edu/courses/18-02sc-multivariable-calculus-fall-2010/): The next course: multivariable calculus.

Glossary

Arc length
Length of a curve, ∫√(1 + f'²) dx.

Carrying capacity
Long-run limit K of a logistic population.

Center of mass
Balance point; moment divided by mass.

Conditional convergence
Converges, but the absolute values diverge.

Differential equation
Equation involving derivatives of an unknown function.

Eccentricity
How stretched an orbit is.

Euler's method
Step-by-step numerical solution of a DE.

Geometric series
Σar^n.

Improper integral
Integral with an infinite limit or integrand.

Integration by parts
$\int u\,dv = uv - \int v\,du$.

Logistic equation
$P' = rP(1 - P/K)$.

Maclaurin series
Taylor series at 0.

Parametric equations
x and y given as functions of t.

Partial fractions
Splitting a rational function into simpler terms.

Polar coordinates
Position given by distance r and angle θ.

Power series
$\Sigma c_n(x - a)^n$.

Radius of convergence
Distance from the center within which a power series converges.

Separable equation
DE that can be written $g(y)\,dy = h(x)\,dx$.

Simpson's rule
Numerical integration with parabolas.

Taylor series
$\Sigma f^{(n)}(a)(x - a)^n/n!$.

Self-assessment rubric

No grade or credit is given. Use this to judge honestly where you are in each unit.

Level	What it looks like
4 · Mastery	Chooses the right technique or test unprompted, justifies it, and checks results numerically.
3 · Proficient	Solves the practice problems with occasional slips.
2 · Developing	Follows worked examples but struggles to choose a method.
1 · Beginning	Applies formulas without checking conditions; needs to revisit the unit.

Suggested weighting if you want a score

Part	Weight
Practice problems	30%
Computer labs	15%
Midterm practice set	25%
Final practice set	30%

Aim for level 3 or better in every unit before moving to the next course.

Related courses

- Calculus I MATH 110
- Intro to Aerospace Engineering AERO 101
- General Chemistry I + Lab CHEM 110 + CHEM 110L
- All courses