Last updated: November 1, 2026

🗺️ Four-year roadmap · Year 2, Fall

About this course

This free course has 8 units and 4 labs. Those are the real counts, not a longer syllabus cut short.

A check is one number or code result on a course page that a script recomputes before the page is published. The printed number and the recomputed number have to match.

> This is a free self-study course outline written by Saving Planets. It is not a university course, it carries no academic credit, and no school is involved.
>
> Its scope follows third-semester college calculus, modeled on the topics and depth of open materials such as OpenStax Calculus Volume 3 and MIT OpenCourseWare 18.02 Multivariable Calculus, and on the syllabi of courses such as Stanford Math 51/52 and Georgia Tech MATH 2551. We link to the free materials throughout. None of those organizations endorses this page.
>
> What we add: every unit connects to real wind, ocean, terrain, climate and satellite problems with a cited public source, every number on the page is re-computed by a checking script before it goes live, and four computer labs apply the ideas to wind fields, elevation grids, lake volumes and orbits.
>
> Work through it in order. Each unit has goals, key ideas, equations, step-by-step examples, practice with full answers, common mistakes, flashcards and a quiz. Your progress is saved only in this browser.

How this course helps save the planet

The planet is three-dimensional and always moving. Winds and ocean currents are vector fields, terrain and temperature are functions of two or three variables, heat and pollutants cross surfaces as flux, and the satellites that measure all of this follow vector-valued paths in space.

Each unit ends with a short, sourced section showing how its ideas help both people and nature.

For instructors

You can teach this course, or let a student work through it alone. It is free. Saving Planets does not give a grade or school credit.

- This page is the whole course: 8 units and 4 labs (MATH 210).
- Plan on 15 weeks, about 6 to 9 hours a week, including the lab weeks.
- Follow the 15-week schedule. Read the unit, do the practice on paper, then open the answer. In a lab week, do the lab or the home version.
- A check is one number or code result on the page that a script recomputes before we publish. The number you see and the recomputed number have to match.
- Use Print this page, or download the PDF study guide. The formula sheet is near the end.
- Ask: how does this unit help people and the planet? Each unit has a green box with a sourced fact.
- Copy the teacher guide template and write in your class name, the time you have, and the questions you will ask. Open the teacher guide template

Before you start

Prerequisites

- Calculus I and II: derivatives, integrals, substitution and integration by parts, parametric and polar curves.
- Basic trigonometry and the equation of a circle and sphere.
- Physics I helps for the work, flux and orbit examples but is not required.

How to study

- Draw a 3D picture for every problem: axes, the region, the vectors.
- Do every practice problem on paper before opening the answer.
- Check units and signs: a flux, work or circulation should make physical sense.
- For multiple integrals, sketch the region and write the limits before integrating anything.
- Use a 3D grapher (Desmos 3D or GeoGebra 3D) to see surfaces and fields.
- Retake the quizzes a few days later. Spaced practice beats one long session.

What you need

- A computer or tablet with a spreadsheet.
- A free 3D grapher such as Desmos 3D or GeoGebra 3D.
- Optional: Python with NumPy (free in Google Colab).

Suggested 15-week schedule

About 6 to 9 hours a week: reading, practice problems and, in lab weeks, the lab.

Week	Unit	Topics	Reading
1	Vectors and the Geometry of Space	3D coordinates, vectors, dot product	OpenStax 2.1–2.3

Week	Unit	Topics	Reading
2	Vectors and the Geometry of Space	Cross product, lines and planes	OpenStax 2.4–2.5
3	Vector Functions and Motion in Space	Vector functions, motion in space, curvature; Lab D	OpenStax 3.1–3.4
4	Partial Derivatives	Functions of several variables, limits, partial derivatives	OpenStax 4.1–4.3
5	Partial Derivatives	Tangent planes, linear approximation, chain rule	OpenStax 4.4–4.5
6	Gradients and Optimization	Gradient, directional derivatives, extrema	OpenStax 4.6–4.7
7	Gradients and Optimization	Lagrange multipliers; Lab B; midterm practice	OpenStax 4.8
8	Double Integrals	Double integrals over rectangles and general regions	OpenStax 5.1–5.2
9	Double Integrals	Polar double integrals; Lab C	OpenStax 5.3
10	Triple Integrals and Change of Variables	Triple integrals; cylindrical and spherical coordinates	OpenStax 5.4–5.5
11	Triple Integrals and Change of Variables	Mass and center of mass; change of variables	OpenStax 5.6–5.7
12	Vector Fields, Line Integrals and Green’s Theorem	Vector fields, line integrals, conservative fields, Green’s theorem	OpenStax 6.1–6.4
13	Surface Integrals, Stokes’ and the Divergence Theorem	Divergence and curl; surface integrals and flux	OpenStax 6.5–6.6
14	Surface Integrals, Stokes’ and the Divergence Theorem	Stokes’ and Divergence theorems; Lab A	OpenStax 6.7–6.8
15	Review	Review and final practice set	All units

Your progress

Tick units and labs as you finish them, and take the quizzes. Progress and quiz scores are saved only in this browser, never sent to us.

Units

Vectors and the Geometry of Space

3D coordinates, vectors, the dot and cross products, and equations of lines and planes.

Learning objectives

By the end of this unit you can:

- Add, scale and find the length of vectors in 3D.
- Use the dot product for angles and projections.
- Use the cross product for normals, areas and volumes.
- Write equations of lines and planes, and find distances.

Key concepts

- A vector has magnitude and direction; in components, v = ⟨v_1, v_2, v_3⟩ and |v| = √(v_1^2 + v_2^2 + v_3^2).
- The dot product a · b = |a||b| cos θ measures how much two vectors point the same way. It is zero exactly when they are perpendicular.
- The cross product a × b is perpendicular to both, with length equal to the area of the parallelogram they span.
- A plane is fixed by a point and a normal vector n: n · (r − r_0) = 0. A line is a point plus multiples of a direction: r = r_0 + t v.

Key equations

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

a· b=| a|| b|θ

Dot product and angle

$$\operatorname{proj}_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$$

_ b a=(a· b)/(| b|^2) b

Projection

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

| a× b|=| a|| b|θ

Cross product length = parallelogram area

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

`a(x-x_0)+b(y-y_0)+c(z-z_0)=0`

Plane with normal ⟨a, b, c⟩

$$D = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

`D=|ax_1+by_1+cz_1-d|^2+b^2+c^2`

Distance from a point to the plane ax + by + cz = d

Worked examples, step by step

Example 1.1 For a = ⟨1, 2, 2⟩ and b = ⟨2, −1, 2⟩, find |a|, a · b and the angle between them.

1. |a| = √(1 + 4 + 4) = 3, and |b| = 3 too.
2. a · b = 2 − 2 + 4 = 4.
3. cos θ = 4/9, so θ = arccos(0.444).

Answer: |a| = 3, a · b = 4, θ ≈ 63.6°.

Example 1.2 Find a × b for the same vectors and the area of the parallelogram they span.

1. a × b = ⟨2·2 − 2·(−1), 2·2 − 1·2, 1·(−1) − 2·2⟩.
2. = ⟨6, 2, −5⟩. Check: it dots to 0 with a and with b.
3. Area = |⟨6, 2, −5⟩| = √65.

Answer: ⟨6, 2, −5⟩; area √65 ≈ 8.06.

Example 1.3 A wind of 10 m/s blows at 30° to the axis of a wind turbine's rotor. What fraction of the head-on power does the turbine see, if power grows with the cube of the wind speed along its axis?

1. Component along the axis: 10 cos 30° = 8.66 m/s (a dot product with the unit axis vector).
2. Power ratio = (cos 30°)³.

Answer: About 0.65: a 30° misalignment loses roughly a third of the power, which is why turbines yaw to face the wind.

Practice problems

Work each on paper first, then check.

1. Find the plane through (1, 0, 0), (0, 2, 0) and (0, 0, 3).

   Intercept form x/1 + y/2 + z/3 = 1, i.e. 6x + 3y + 2z = 6.

2. Find the distance from the origin to that plane.

   |−6|/√(36 + 9 + 4) = 6/7 ≈ 0.857.

3. Find the projection of ⟨3, 4, 0⟩ onto ⟨1, 0, 0⟩.

   ⟨3, 0, 0⟩.

4. Find the volume of the box (parallelepiped) spanned by ⟨1, 0, 0⟩, ⟨0, 2, 0⟩ and ⟨1, 1, 3⟩.

   |a · (b × c)| = det = 1·(2·3 − 0·1) = 6.

5. Are ⟨1, 2, 3⟩ and ⟨3, 0, −1⟩ perpendicular?

   Dot product 3 + 0 − 3 = 0, so yes.

Common misconceptions

Myth: a × b = b × a.
Actually: The cross product is anti-commutative: b × a = −(a × b).

Myth: The dot product of two vectors is a vector.
Actually: It is a scalar (a number).

Myth: A plane's equation tells you its direction.
Actually: Its coefficients give the normal, which is perpendicular to the plane.

Flashcards

Dot product

a · b = |a||b| cos θ; zero when perpendicular.

Cross product

Perpendicular to both; length = parallelogram area.

Projection

(a · b/|b|²) b.

Triple product

|a · (b × c)| = box volume.

Plane

Line

$n \cdot (r - r_0) = 0$. $r = r_0 + t\,v$.

Self-check quiz

1. If a · b = 0 (both nonzero), the vectors are…

parallel

perpendicular

equal

opposite

perpendicular. cos θ = 0, so θ = 90°.

2. The length of a × b equals…

the angle between them

the parallelogram’s area

a · b

|a| + |b|

the parallelogram’s area. |a||b| sin θ.

3. The normal to 2x − y + 5z = 7 is…

⟨2, −1, 5⟩

⟨7, 0, 0⟩

⟨2, 1, 5⟩

⟨1, 1, 1⟩

⟨2, −1, 5⟩. Read off the coefficients.

4. |⟨2, 3, 6⟩| =

11

7

√11

49

7. √(4 + 9 + 36) = √49.

Try this quiz again

How this helps save the planet

Wind is a vector, and turbines turn their rotors to face it because power depends on the cube of the component along the rotor axis, a dot product in disguise.

The same arithmetic tells a solar panel how much sunlight it catches: the useful part is the dot product of the sunlight direction with the panel's normal vector.

On Saving Planets: Physics I, Unit 3: wind power

Sources: U.S. DOE: How do wind turbines work? (https://www.energy.gov/eere/wind/how-do-wind-turbines-work) · OpenStax Calculus 3, 2.3 The Dot Product (https://openstax.org/books/calculus-volume-3/pages/2-3-the-dot-product)

Free learning resources

- OpenStax Calculus 3, 2.3 The Dot Product (https://openstax.org/books/calculus-volume-3/pages/2-3-the-dot-product): Angles and projections.
- OpenStax Calculus 3, 2.4 The Cross Product (https://openstax.org/books/calculus-volume-3/pages/2-4-the-cross-product): Normals, areas and torque.
- OpenStax Calculus 3, 2.5 Equations of Lines and Planes in Space (https://openstax.org/books/calculus-volume-3/pages/2-5-equations-of-lines-and-planes-in-space): Lines, planes and distances.

Watch the lecture

Session 79 Clip: Vector Fields in Space (https://ocw.mit.edu/courses/18-02sc-multivariable-calculus-fall-2010/resources/clip-vector-fields-in-space/) (in English)

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☐ **I finished this unit**

UNIT 2 · WEEK 3

Vector Functions and Motion in Space

Curves in space as vector-valued functions: velocity, acceleration, arc length, curvature, and the motion of satellites.

Learning objectives

By the end of this unit you can:

- Differentiate and integrate vector-valued functions.
- Find velocity, speed and acceleration along a path.
- Compute arc length and curvature.
- Analyze circular orbits with vector calculus.

Key concepts

- A space curve is r(t) = ⟨x(t), y(t), z(t)⟩; differentiate component by component.
- Velocity v = r'(t) is tangent to the path; speed is |v|; acceleration is a = v'.
- Arc length is the integral of speed: L = ∫|r'(t)| dt.
- In uniform circular motion the speed is constant but the acceleration points to the center with size v²/r.

Key equations

$$\mathbf{v} = \mathbf{r}'(t),\ \mathbf{a} = \mathbf{r}''(t)$$

`v= r'(t),\ a= r''(t)`

Velocity and acceleration

$$L = \int_a^b |\mathbf{r}'(t)|\,dt$$

`L=_a^b| r'(t)| dt`

Arc length

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$$

`=(| r'× r''|)/(| r'|^3)`

Curvature

$$v = \sqrt{GM/r},\ T = 2\pi r/v$$

$v=/r,\ T=2\pi\ r/v$

Circular orbit speed and period

Worked examples, step by step

Example 2.1 Find the length of one turn of the helix $r(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 2\pi$.

1. $r' = \langle -\sin t, \cos t, 1 \rangle$, speed $\sqrt{(\sin^2 t + \cos^2 t + 1)} = \sqrt{2}$.
2. $L = \int_0^{(2\pi)} \sqrt{2}\, dt = 2\pi\sqrt{2}$.

Answer: $2\pi\sqrt{2} \approx 8.89$.

Example 2.2 Landsat 9 orbits 705 km above Earth (mean radius 6,371 km). Treating the orbit as a circle, find its speed, acceleration and period.

1. $r = 7{,}076$ km $= 7.076 \times 10^6$ m; $GM = 3.986 \times 10^{14}$ m³/s².
2. $v = \sqrt{(GM/r)} = 7{,}505$ m/s.
3. $a = v^2/r = 7.96$ m/s², pointing to Earth's center.
4. $T = 2\pi r/v = 5{,}924$ s.

Answer: About 7.5 km/s and 98.7 minutes per orbit, matching NASA's listed 99-minute period.

Example 2.3 Find the curvature of a circle of radius 2, $r(t) = \langle 2\cos t, 2\sin t, 0 \rangle$.

1. $r' = \langle -2\sin t, 2\cos t, 0 \rangle$, $|r'| = 2$.
2. $r'' = \langle -2\cos t, -2\sin t, 0 \rangle$; $r' \times r'' = \langle 0, 0, 4 \rangle$.
3. $\kappa = 4/2^3$.

Answer: $\kappa = 1/2$: a circle's curvature is 1/radius.

Practice problems

Work each on paper first, then check.

1. A projectile has $r(t) = \langle 20t, 20t - 4.9t^2 \rangle$ m. Find its velocity and speed at $t = 1$ s.

   $v = \langle 20, 20 - 9.8t \rangle = \langle 20, 10.2 \rangle$; speed $\sqrt{(400 + 104.04)} \approx 22.45$ m/s.

2. Find the acceleration of $r(t) = \langle t^2, 3t, e^t \rangle$.

   $\langle 2, 0, e^t \rangle$.

3. Find the length of $r(t) = \langle 3t, 4t, 0\rangle$ for $0 \le t \le 2$.

   Speed 5, so $L = 10$.

4. A satellite orbits at 35,786 km altitude. Find its period.

   $r = 42{,}157$ km; $T = 2\pi\sqrt{(r^3/GM)} \approx 86{,}160$ s ≈ 23.93 h: one sidereal day, so it is geostationary.

Common misconceptions

Myth: Constant speed means zero acceleration.

Actually: Turning is acceleration: a satellite in a circular orbit accelerates toward Earth the whole time.

Myth: Arc length is $|r(b) - r(a)|$.

Actually: That is the straight-line distance; arc length integrates speed along the path.

Myth: Higher satellites move faster.

Actually: $v = \sqrt{(GM/r)}$ falls with distance; higher orbits are slower and longer.

Flashcards

Vector function

$r(t) = \langle x(t), y(t), z(t)\rangle$.

Velocity

$r'(t)$, tangent to the path.

Speed

$|r'(t)|$.

Arc length

$\int |r'|\, dt$.

Curvature

$|r' \times r''|/|r'|^3$.

Circular orbit

$v = \sqrt{(GM/r)}$.

Self-check quiz

1. Velocity is...

$r(t)$

$r'(t)$

|r(t)|

∫r dt

r'(t). The derivative of position.

2. Arc length integrates…

position

speed

acceleration

curvature

speed. L = ∫|r'| dt.

3. Curvature of a circle of radius 5 is…

5

1/5

25

0

1/5. κ = 1/R.

4. Doubling a circular orbit’s radius changes its speed by a factor…

2

1/√2

1/2

√2

1/√2. v ∝ 1/√r.

Try this quiz again

How this helps save the planet

Earth-observing satellites such as Landsat 9 follow vector paths chosen with exactly these formulas. Its near-polar, sun-synchronous orbit at 705 km lets it image the same place every 16 days at about the same local time, so changes in forests, glaciers and cities stand out.

On Saving Planets: Landsat 9 · Calculus II, Unit 7: polar orbits

Sources: NASA: Landsat 9 quick facts (https://landsat.gsfc.nasa.gov/satellites/landsat-9/) · NASA Earth fact sheet (https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html)

Free learning resources

- OpenStax Calculus 3, 3.2 Calculus of Vector-Valued Functions (https://openstax.org/books/calculus-volume-3/pages/3-2-calculus-of-vector-valued-functions): Derivatives and integrals.
- OpenStax Calculus 3, 3.4 Motion in Space (https://openstax.org/books/calculus-volume-3/pages/3-4-motion-in-space): Velocity, acceleration and Kepler's laws.

Watch the lecture

Session 79 Clip: Vector Fields in Space (https://ocw.mit.edu/courses/18-02sc-multivariable-calculus-fall-2010/resources/clip-vector-fields-in-space/) (in English)

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☐ I finished this unit

Partial Derivatives

Functions of several variables: level curves, partial derivatives, tangent planes, linear approximation and the multivariable chain rule.

Learning objectives

By the end of this unit you can:

- Read level curves and contour maps.
- Compute partial derivatives and interpret them as rates.
- Find tangent planes and use linear approximation.
- Apply the multivariable chain rule.

Key concepts

- A function z = f(x, y) is a surface; its level curves f = c are a contour map.
- ∂f/∂x is the rate of change in x with y held fixed.
- Near a point, a smooth surface looks like its tangent plane: Δf ≈ f_xΔx + f_yΔy.
- Chain rule: if x and y depend on t, df/dt = f_x dx/dt + f_y dy/dt.

Key equations

$$f_x = \frac{\partial f}{\partial x}$$

`f_x=( f)/( x)`

Partial derivative

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

`z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)`

Tangent plane

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

`df/dt=f_xdx/dt+f_ydy/dt`

Chain rule

$$f_{xy} = f_{yx}$$

`f_xy=f_yx`

Clairaut's theorem (smooth f)

Worked examples, step by step

Example 3.1 For $f(x, y) = x^2y + y^3$, find f_x and f_y at (1, 2), and use the tangent plane to estimate f(1.1, 1.9).

1. $f_x = 2xy = 4$; $f_y = x^2 + 3y^2 = 13$; $f(1, 2) = 10$.
2. $f \approx 10 + 4(0.1) + 13(-0.1) = 9.1$.
3. True value: $1.21 \times 1.9 + 6.859 = 9.158$.

Answer: Estimate 9.1 versus the true 9.158.

Example 3.2 For an ideal gas, $P = nRT/V$. With n = 1 mol, T = 300 K and V = 0.025 m^3, find P, $\partial P/\partial T$ and $\partial P/\partial V$.

1. $P = 8.314 \times 300/0.025 = 99{,}768$ Pa.
2. $\partial P/\partial T = nR/V = 332.6$ Pa/K.
3. $\partial P/\partial V = -nRT/V^2 = -P/V = -3.99 \times 10^6$ Pa/m^3.

Answer: P ≈ 99.8 kPa; warming 1 K raises it about 333 Pa; expanding lowers it.

Example 3.3 A drone flies with $x = 2t$, $y = t^2$. The temperature it measures is $T = 20 + 0.1x - 0.05y$ (°C, km). Find dT/dt at t = 1.

1. $T_x = 0.1$, $T_y = -0.05$; $dx/dt = 2$, $dy/dt = 2t = 2$.
2. $dT/dt = 0.1(2) - 0.05(2)$.

Answer: 0.1 °C per unit time.

Practice problems

Work each on paper first, then check.

1. Find f_x and f_y for $f = e^{(xy)}$.

   $f_x = y\, e^{(xy)}$, $f_y = x\, e^{(xy)}$.

2. Find f_{xy} for $f = x^3y^2$.

   $f_x = 3x^2y^2$, $f_{xy} = 6x^2y$ (same as f_{yx}).

3. Find the tangent plane to $z = x^2 + y^2$ at (1, 1, 2).

   $z = 2 + 2(x - 1) + 2(y - 1)$, i.e. $z = 2x + 2y - 2$.

4. Estimate $\sqrt{(3.02^2 + 3.97^2)}$ by linear approximation at (3, 4).

   $f = 5$, $f_x = 3/5$, $f_y = 4/5$: $5 + 0.6(0.02) + 0.8(-0.03) = 4.988$ (true 4.9880).

Common misconceptions

Myth: $\partial f/\partial x$ ignores y.

Actually: It holds y fixed; y can still appear in the answer.

Myth: If both partials exist, f is smooth.

Actually: Partials can exist where f is not even continuous; differentiability needs more.

Myth: Linear approximation is good anywhere.

Actually: Only near the base point.

Flashcards

Partial derivative

Rate of change in one variable with the others fixed.

Level curve

f(x, y) = c; a contour line.

Tangent plane

$z = z_0 + f_x\Delta x + f_y\Delta y$.

Linear approximation

$\Delta f \approx f_x\Delta x + f_y\Delta y$.

Chain rule

$df/dt = f_x x' + f_y y'$.

Clairaut

$f_{xy} = f_{yx}$ for smooth f.

Self-check quiz

1. For $f = x^2y$, f_x =

2x

2xy

x^2

x^2y

2xy. Treat y as a constant.

2. Level curves of f(x, y) = $x^2 + y^2$ are...

lines

circles

parabolas

points

circles. $x^2 + y^2 = c$.

3. For smooth f, f_xy and f_yx are...

opposite

equal

unrelated

zero

equal. Clairaut's theorem.

4. df/dt for f(x(t), y(t)) is...

f_x + f_y

f_xx' + f_yy'

f_xf_y

x'y'

f_xx' + f_yy'. Multivariable chain rule.

Try this quiz again

How this helps save the planet

Weather maps are contour maps of functions of two variables: isobars are level curves of pressure, isotherms of temperature. Partial derivatives tell forecasters how fast pressure or temperature changes eastward or northward, which drives the winds.

On Saving Planets: Calculus I

Sources: NOAA/NWS JetStream: Wind (https://www.weather.gov/jetstream/wind) · OpenStax Calculus 3, 4.3 Partial Derivatives (https://openstax.org/books/calculus-volume-3/pages/4-3-partial-derivatives)

Free learning resources

- OpenStax Calculus 3, 4.3 Partial Derivatives (https://openstax.org/books/calculus-volume-3/pages/4-3-partial-derivatives): Free textbook section.
- OpenStax Calculus 3, 4.4 Tangent Planes and Linear Approximations (https://openstax.org/books/calculus-volume-3/pages/4-4-tangent-planes-and-linear-approximations): Tangent planes.
- OpenStax Calculus 3, 4.5 The Chain Rule (https://openstax.org/books/calculus-volume-3/pages/4-5-the-chain-rule): Chain rule.

Watch the lecture

Session 26 Clip: Partial Derivatives (https://ocw.mit.edu/courses/18-02sc-multivariable-calculus-fall-2010/resources/clip-partial-derivatives/) (in English)

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☐ I finished this unit

Gradients and Optimization

The gradient vector, directional derivatives, critical points and the second-derivative test, and constrained optimization with Lagrange multipliers.

Learning objectives

By the end of this unit you can:

- Compute the gradient and directional derivatives.
- Explain why the gradient points uphill and is perpendicular to level curves.
- Classify critical points with the second-derivative test.
- Solve constrained problems with Lagrange multipliers.

Key concepts

- ∇f = ⟨f_x, f_y⟩ points in the direction of fastest increase, and its length is that fastest rate.
- The rate of change in the direction of a unit vector u is D_u f = ∇f · u.
- The gradient is perpendicular to level curves, so water flows downhill along −∇h, across the contours.
- At an interior extremum ∇f = 0. The test D = f_xx f_yy − f_{xy}^2 decides: D > 0 and f_xx > 0 is a minimum; D < 0 is a saddle.
- To optimize f subject to g = c, solve ∇f = λ∇g: at the best point the level curve of f just touches the constraint.

Key equations

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

```
f= f_x,f_y,f_z
```

Gradient

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$$

```
D_ uf= f· u
```

Directional derivative

$$D = f_{xx} f_{yy} - f_{xy}^2$$

```
D=f_xxf_yy-f_xy^2
```

Second-derivative test

$$\nabla f = \lambda \nabla g, \quad g = c$$

`f= g, g=c`

Lagrange multiplier condition

Worked examples, step by step

Example 4.1 A hill has height h(x, y) = 500 − 0.01x² − 0.02y² (meters, x and y in meters). At (100, 50), find ∇h, the steepest slope, and the slope heading toward ⟨3, 4⟩/5.

1. ∇h = ⟨−0.02x, −0.04y⟩ = ⟨−2, −2⟩.
2. Steepest uphill rate |∇h| = 2√2 = 2.83 m per m.
3. D_u h = ⟨−2, −2⟩ · ⟨0.6, 0.8⟩ = −2.8.

Answer: Steepest climb 2.83 (uphill toward ⟨−1, −1⟩); heading ⟨3, 4⟩ you descend 2.8 m per m. Rain runs off along −∇h.

Example 4.2 Find and classify the critical point of f = x² + xy + y² − 3x.

1. f_x = 2x + y − 3 = 0, f_y = x + 2y = 0 give x = 2, y = −1.
2. f_xx = 2, f_yy = 2, f_xy = 1: D = 3 > 0, f_xx > 0.
3. f(2, −1) = 4 − 2 + 1 − 6 = −3.

Answer: A minimum, f = −3 at (2, −1).

Example 4.3 A can must hold 1 liter (1,000 cm³). Which radius and height use the least metal (closed cylinder)?

1. Minimize A = 2πr² + 2πrh with g = πr²h = 1,000.
2. ∇A = λ∇g gives 4πr + 2πh = λ2πrh and 2πr = λπr², so λ = 2/r and h = 2r.
3. πr²(2r) = 1,000 gives r = (500/π)^(1/3).

Answer: r ≈ 5.42 cm, h ≈ 10.84 cm (height = diameter), area ≈ 554 cm².

Practice problems

Work each on paper first, then check.

1. Find ∇f for f = xyz at (1, 2, 3).

$\langle yz, xz, xy \rangle = \langle 6, 3, 2 \rangle$.

2. Find $D_u f$ for $f = x^2 + y^2$ at $(1, 1)$ toward $\langle 1, 0 \rangle$.

   $\nabla f = \langle 2, 2 \rangle$; $D_u f = 2$.

3. Classify the critical point of $f = x^2 - y^2$.

   At $(0, 0)$: $D = (2)(-2) - 0 = -4 < 0$, a saddle.

4. Fence a rectangle along a straight river with 40 m of fence (no fence on the river side). Maximize the area.

   Maximize xy with $x + 2y = 40$: $y = 10$, $x = 20$, area 200 m^2.

5. Maximize $f = x + y$ on the circle $x^2 + y^2 = 2$.

   $\langle 1, 1 \rangle = \lambda \langle 2x, 2y \rangle$ gives $x = y = 1$; maximum 2.

Common misconceptions

Myth: The gradient points along the level curve.

Actually: It is perpendicular to the level curve.

Myth: $\nabla f = 0$ means a maximum or minimum.

Actually: It may be a saddle; use the D test.

Myth: λ is just a trick to eliminate.

Actually: λ is the rate at which the best value changes as the constraint c changes, which is useful in economics and engineering.

Flashcards

Gradient

$\langle f_x, f_y, f_z \rangle$; fastest increase.

Directional derivative

$\nabla f \cdot u$.

Critical point

$\nabla f = 0$.

Saddle point

$D < 0$.

Lagrange condition

$\nabla f = \lambda \nabla g$.

Steepest descent

Move along $-\nabla f$.

Self-check quiz

1. The gradient points...

along level curves

in the direction of fastest increase

downhill

toward the origin

in the direction of fastest increase. Its length is the maximum rate.

2. D < 0 at a critical point means...

minimum

maximum

saddle

no conclusion

saddle. Curving up one way, down another.

3. Lagrange multipliers solve...

$\nabla f = 0$

$\nabla f = \lambda \nabla g$

$f = g$

$f' = \lambda$

$\nabla f = \lambda \nabla g$. Gradients are parallel at a constrained optimum.

4. $D_u f$ is largest when u is...

perpendicular to ∇f

parallel to ∇f

opposite ∇f

any direction

parallel to ∇f. ∇f · u = |∇f| cos θ.

Try this quiz again

How this helps save the planet

Water flows down the gradient of terrain. Flood maps, watershed boundaries and erosion models are built by computing −∇h from elevation data, such as the free lidar elevation models of the USGS 3D Elevation Program.

Lagrange multipliers minimize material for a required volume or strength. Using less material for the same job is a direct cut in mining, energy and emissions.

On Saving Planets: Rivers hub · Engineering Graphics / CAD: lightweighting

Sources: USGS 3D Elevation Program (3DEP) (https://www.usgs.gov/3d-elevation-program) · OpenStax Calculus 3, 4.8 Lagrange Multipliers (https://openstax.org/books/calculus-volume-3/pages/4-8-lagrange-multipliers)

Free learning resources

- OpenStax Calculus 3, 4.6 Directional Derivatives and the Gradient (https://openstax.org/books/calculus-volume-3/pages/4-6-directional-derivatives-and-the-gradient): Gradients.
- OpenStax Calculus 3, 4.7 Maxima/Minima Problems (https://openstax.org/books/calculus-volume-3/pages/4-7-maxima-minima-problems): Second-derivative test.
- OpenStax Calculus 3, 4.8 Lagrange Multipliers (https://openstax.org/books/calculus-volume-3/pages/4-8-lagrange-multipliers): Constrained optimization.

☐ **I finished this unit**

Double Integrals

Integrating over regions in the plane: iterated integrals, Fubini’s theorem, general regions, polar coordinates, area, mass and averages.

Learning objectives

By the end of this unit you can:

- Evaluate double integrals as iterated integrals.
- Set up limits for general regions and reverse the order.
- Convert to polar coordinates with dA = r dr dθ.
- Use double integrals for area, mass, averages and totals over a map.

Key concepts

- ∬_R f dA adds f times tiny areas. Over a rectangle, Fubini says integrate one variable, then the other, in either order.
- For a general region, the inner limits are curves; sketch the region to read them.
- Round regions are easier in polar coordinates, where the area element is r dr dθ (not dr dθ).
- The average of f over R is ∬f dA divided by the area of R.

Key equations

$$\iint_R f\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

```
_Rf dA=_a^b\!_c^df(x,y) dy dx
```

Fubini’s theorem

$$dA = r\,dr\,d\theta$$

```
dA=r dr dθ
```

Polar area element

$$\bar{f} = \frac{1}{\text{Area}(R)} \iint_R f\,dA$$

```
f=1Area(R)_Rf dA
```

Average value

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

_-∞^∞e^-x^2dx=π

Gaussian integral (via polar coordinates)

Worked examples, step by step

Example 5.1 Evaluate ∬_R xy dA over R = [0, 2] × [0, 3].

1. $\int_0^2 x\,dx = 2$ and $\int_0^3 y\,dy = 4.5$.
2. The integral separates: 2 × 4.5.

Answer: 9.

Example 5.2 Rain falls on a 10 km × 5 km catchment with depth d(x, y) = 20 + 2x mm (x in km, across the 10 km side). How much water fell?

1. Total = $\int_0^{10}\int_0^5 (20 + 2x)\,dy\,dx = 5\int_0^{10} (20 + 2x)\,dx$.
2. = 5(200 + 100) = 1,500 mm·km².
3. 1 mm over 1 km² is 1,000 m³.

Answer: 1.5 × 10⁶ m³ (average depth 30 mm). This is how rain gauges and radar become river-flow forecasts.

Example 5.3 Evaluate ∬ e^(−(x² + y²)) dA over the unit disk, then over the whole plane.

1. Polar: $\int_0^{2\pi}\int_0^1 e^{-r^2}\,r\,dr\,d\theta = 2\pi \times \tfrac{1}{2}(1 - e^{-1})$.
2. Over the plane, the r-integral goes to ∞ and gives ½.

Answer: π(1 − 1/e) ≈ 1.986 on the disk; π over the plane, which proves $\int e^{-x^2}\,dx = \sqrt{\pi}$.

Practice problems

Work each on paper first, then check.

1. $\int_0^1\int_0^x x\,dy\,dx$

   $\int_0^1 x^2\,dx = 1/3$.

2. Reverse the order of $\int_0^1\int_y^1 f\,dx\,dy$.

$\int_0^1 \int_0^x f\, dy\, dx$.

3. Find the area of a circle of radius a in polar coordinates.

   $\int_0^{(2\pi)} \int_0^a r\, dr\, d\theta = \pi a^2$.

4. Find the volume under $z = 4 - x^2 - y^2$ above the disk $x^2 + y^2 \le 4$.

   $\int_0^{(2\pi)} \int_0^2 (4 - r^2)\, r\, dr\, d\theta = 2\pi(8 - 4) = 8\pi \approx 25.1$.

5. Average of $f = x + y$ over $[0, 1] \times [0, 1]$.

   $\iint (x + y)\, dA = 1$, area 1, average 1.

Common misconceptions

Myth: In polar, $dA = dr\, d\theta$.

Actually: $dA = r\, dr\, d\theta$; forgetting r is the most common error.

Myth: Inner limits must be numbers.

Actually: For a general region, inner limits are functions of the outer variable.

Myth: Order never matters.

Actually: The value doesn't, but the limits change, and one order is often much easier.

Flashcards

Double integral

$\iint_R f\, dA$.

Fubini

Iterate in either order over a rectangle.

Type I region

$a \le x \le b$, $g_1(x) \le y \le g_2(x)$.

Polar dA

$r\, dr\, d\theta$.

Average value

$\iint f\, dA$ / Area.

Gaussian integral

$\int e^{(-x^2)}\, dx = \sqrt{\pi}$.

Self-check quiz

1. Polar area element:

dr dθ

r dr dθ

r^2 dr dθ

dθ/r

r dr dθ. A polar patch is r dθ by dr.

2. ∬_R 1 dA equals...

0

the area of R

the perimeter

1

the area of R. Adds up the area.

3. $\int_0^1 \int_0^2$ 3 dy dx =

3

5

6

2

6. 3 × area 2.

4. Fubini's theorem lets you...

skip limits

integrate one variable at a time

differentiate

use polar always

integrate one variable at a time. Iterated integrals.

Try this quiz again

How this helps save the planet

Totals over a map are double integrals: rain on a watershed, carbon stored in a forest, or crops harvested from a field. Hydrologists turn rainfall maps into volumes this way to predict floods and plan water supply.

On Saving Planets: Rivers hub · Deforestation hub

Sources: USGS Water Science School: Precipitation and the water cycle (https://www.usgs.gov/special-topics/water-science-school/science/precipitation-and-water-cycle) · OpenStax Calculus 3, 5.2 Double Integrals over General Regions (https://openstax.org/books/calculus-volume-3/pages/5-2-double-integrals-over-general-regions)

Free learning resources

- OpenStax Calculus 3, 5.1 Double Integrals over Rectangular Regions (https://openstax.org/books/calculus-volume-3/pages/5-1-double-integrals-over-rectangular-regions): Fubini.
- OpenStax Calculus 3, 5.2 Double Integrals over General Regions (https://openstax.org/books/calculus-volume-3/pages/5-2-double-integrals-over-general-regions): General regions.
- OpenStax Calculus 3, 5.3 Double Integrals in Polar Coordinates (https://openstax.org/books/calculus-volume-3/pages/5-3-double-integrals-in-polar-coordinates): Polar.

Watch the lecture

Session 47 Clip: Definition of Double Integration (https://ocw.mit.edu/courses/18-02sc-multivariable-calculus-fall-2010/resources/clip-definition-of-double-integration/) (in English)

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☐ **I finished this unit**

Triple Integrals and Change of Variables

Integrating over solids in rectangular, cylindrical and spherical coordinates, and the Jacobian for general changes of variables.

Learning objectives

By the end of this unit you can:

- Set up and evaluate triple integrals.
- Use cylindrical (r dz dr dθ) and spherical (ρ² sin φ dρ dφ dθ) coordinates.
- Compute the Jacobian of a change of variables.
- Find volumes, masses and centers of mass of solids.

Key concepts

- ∭_E f dV adds f over tiny boxes; with f = density it gives mass.
- Cylindrical coordinates suit pipes, tanks and cones: dV = r dz dr dθ.
- Spherical coordinates suit balls and planets: dV = ρ² sin φ dρ dφ dθ.
- Any substitution x = x(u, v) stretches area by the Jacobian |∂(x, y)/∂(u, v)|.

Key equations

$$dV = r\,dz\,dr\,d\theta$$

`dV=r dz dr dθ`

Cylindrical volume element

$$dV = \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$$

`dV=ρ^2 dρ d dθ`

Spherical volume element

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = |x_u y_v - x_v y_u|$$

`J=|((x,y))/((u,v))|=|x_uy_v-x_vy_u|`

Jacobian

$$\iint_R f\,dx\,dy = \iint_S f\,|J|\,du\,dv$$

`_Rf dx dy=_Sf |J| du dv`

Worked examples, step by step

Example 6.1 Use spherical coordinates to find the volume of a ball of radius R.

1. $V = \int_0^{(2\pi)}\int_0^{\pi}\int_0^R \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$.
2. $= (R^3/3)(2)(2\pi)$.

Answer: $4\pi R^3/3$.

Example 6.2 Estimate the mass of Earth's atmosphere: the weight of air above each square meter is the surface pressure, so $M = \iint p/g \, dA$ over the sphere.

1. The surface integral of a constant is p/g times the area $4\pi R^2$.
2. $4\pi(6.371 \times 10^6)^2 = 5.10 \times 10^{14}$ m².
3. $M = 101{,}400 \times 5.10 \times 10^{14} / 9.81$.

Answer: About 5.3×10^{18} kg, close to NASA's 5.1×10^{18} kg (land above sea level holds up less air).

Example 6.3 Find the area of the ellipse $x^2/a^2 + y^2/b^2 \le 1$ with $x = au$, $y = bv$.

1. The ellipse becomes the unit disk $u^2 + v^2 \le 1$.
2. $J = ab$.
3. Area $= ab \times \pi$.

Answer: πab.

Practice problems

Work each on paper first, then check.

1. Find the volume of the cone $0 \le z \le 1 - r$ (radius 1, height 1) in cylindrical coordinates.

   $\int_0^{(2\pi)}\int_0^1 (1 - r)\, r \, dr \, d\theta = 2\pi(1/2 - 1/3) = \pi/3$.

2. Evaluate $\iiint 1 \, dV$ over the box $[0, 1] \times [0, 2] \times [0, 3]$.

   6.

3. Find the Jacobian of polar coordinates.

$x_r y_\theta - x_\theta y_r = \cos\theta\,(r\cos\theta) + r\sin\theta\,(\sin\theta) = r$.

4. A cylinder of radius 1 m and height 2 m has density $1{,}000(1 - z/4)$ kg/m³. Find its mass.

   $\pi \int_0^2 1{,}000(1 - z/4)\,dz = \pi \times 1{,}000 \times 1.5 = 1{,}500\pi \approx 4{,}712$ kg.

Common misconceptions

Myth: In spherical coordinates $dV = d\rho\,d\varphi\,d\theta$.

Actually: $dV = \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$.

Myth: φ runs from 0 to 2π.

Actually: φ (from the north pole) runs 0 to π; θ runs 0 to 2π.

Myth: The Jacobian can be negative inside the integral.

Actually: Use its absolute value.

Flashcards

Triple integral

$\iiint_E f\,dV$.

Cylindrical

$x = r\cos\theta$, $y = r\sin\theta$, z; $dV = r\,dz\,dr\,d\theta$.

Spherical

$dV = \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$.

Jacobian

Area or volume stretch factor of a substitution.

Mass

$\iiint$ density dV.

Center of mass

$\iiint x\,\rho\,dV$ / mass.

Self-check quiz

1. The spherical volume element is...

$d\rho\,d\varphi\,d\theta$

$\rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$

$\rho\,d\rho\,d\theta$

r dz dr dθ

$\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$. A spherical box of sides $d\rho$, $\rho \, d\varphi$, $\rho \sin\varphi \, d\theta$.

2. Cylindrical coordinates are best for…

cubes

pipes and tanks

pyramids

planes

pipes and tanks. Circular cross-sections.

3. The Jacobian of x = 2u, y = 3v is…

5

6

1

2/3

6. 2 × 3.

4. Volume of a ball of radius 3:

36π

27π

12π

9π

36π. $4\pi(27)/3$.

Try this quiz again

How this helps save the planet

Climate scientists add up quantities through the volume of the ocean and atmosphere with exactly these integrals. Ocean heat content, the heat stored in the top 2,000 m of the sea, is a triple integral of temperature change times density and heat capacity, and it is one of the clearest records of global warming.

On Saving Planets: Oceans hub

Sources: NOAA NCEI: Global ocean heat content (https://www.ncei.noaa.gov/access/global-ocean-heat-content/) · NASA Earth fact sheet (https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html)

Free learning resources

- OpenStax Calculus 3, 5.4 Triple Integrals (https://openstax.org/books/calculus-volume-3/pages/5-4-triple-integrals): Free textbook section.
- OpenStax Calculus 3, 5.5 Triple Integrals in Cylindrical and Spherical Coordinates (https://openstax.org/books/calculus-volume-3/pages/5-5-triple-integrals-in-cylindrical-and-spherical-coordinates): Cylindrical and spherical.
- OpenStax Calculus 3, 5.7 Change of Variables in Multiple Integrals (https://openstax.org/books/calculus-volume-3/pages/5-7-change-of-variables-in-multiple-integrals): Jacobians.

Watch the lecture

Session 53 Clip: Change of Variables (https://ocw.mit.edu/courses/18-02sc-multivariable-calculus-fall-2010/resources/clip-change-of-variables/) (in English)

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☐ **I finished this unit**

Vector Fields, Line Integrals and Green’s Theorem

Vector fields such as winds and currents, work and circulation along curves, conservative fields and potentials, and Green’s theorem in the plane.

Learning objectives

By the end of this unit you can:

- Sketch and interpret vector fields.
- Evaluate line integrals of scalar functions and of vector fields (work).
- Test whether a field is conservative and find a potential.
- Apply Green’s theorem to circulation and flux in the plane.

Key concepts

- A vector field assigns a vector to every point, like a wind or current map.
- Work along a path is ∫_C F · dr; for a closed loop it is called circulation.
- A conservative field is a gradient, F = ∇φ; its work depends only on the endpoints, and it has zero circulation.
- Green’s theorem: circulation around a closed curve equals the integral of the curl (Q_x − P_y) over the region inside.

Key equations

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

```
_C F· d r=_a^b F( r(t))· r'(t) dt
```

Line integral (work)

$$\int_C \nabla\varphi \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$$

```
_C· d r=(B)-(A)
```

Fundamental theorem for line integrals

$$\oint_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$$

```
_CP dx+Q dy=_D(Q_x-P_y) dA
```

Green’s theorem

$$\text{Area} = \tfrac{1}{2}\oint_C x\,dy - y\,dx$$

```
Area=12_Cx dy-y dx
```

Area from the boundary

Worked examples, step by step

Example 7.1 F = ⟨y, x⟩. Find the work from (0, 0) to (2, 3) along any path.

1. P_y = 1 = Q_x, so F is conservative.
2. φ = xy has ∇φ = ⟨y, x⟩.
3. Work = φ(2, 3) − φ(0, 0).

Answer: 6, on every path.

Example 7.2 A gyre-like current F = ⟨−y, x⟩ circles the origin. Find its circulation around the unit circle directly and by Green's theorem.

1. r = ⟨cos t, sin t⟩, F · r' = sin²t + cos²t = 1, so ∮ = 2π.
2. Green: Q_x − P_y = 1 − (−1) = 2; ∬2 dA = 2 × π.

Answer: 2π ≈ 6.28 both ways.

Example 7.3 Use ½∮(x dy − y dx) to find the area inside x = 3 cos t, y = 2 sin t.

1. x dy − y dx = (3 cos t)(2 cos t) + (2 sin t)(3 sin t) = 6 dt.
2. ½ ∫₀^(2π) 6 dt.

Answer: 6π ≈ 18.85 (πab with a = 3, b = 2).

Practice problems

Work each on paper first, then check.

1. Is F = ⟨2xy, x² + 1⟩ conservative? Find φ.

   P_y = 2x = Q_x; φ = x²y + y.

2. Find ∫_C F · dr for F = ⟨x, y⟩ along y = x² from (0, 0) to (1, 1).

Conservative, $\varphi = (x^2 + y^2)/2$: work 1.

3. Find $\oint(x^2\,dx + xy\,dy)$ around the square $[0, 1] \times [0, 1]$, counterclockwise.

   Green: $\iint y\,dA = 1/2$.

4. Find the circulation of $F = \langle -y, x\rangle$ around a circle of radius 3.

   $2 \times$ area $= 18\pi \approx 56.5$.

Common misconceptions

Myth: Every field with zero curl is conservative.

Actually: True only on simply connected regions (no holes).

Myth: The work along a closed path is always zero.

Actually: Only for conservative fields; a gyre circulates.

Myth: Green's theorem works for any orientation.

Actually: It assumes counterclockwise; clockwise flips the sign.

Flashcards

Vector field

A vector at every point.

Line integral

$\int F \cdot dr$, work along a curve.

Circulation

Line integral around a closed curve.

Conservative

$F = \nabla\varphi$; path-independent.

Potential

φ with $\nabla\varphi = F$.

Green's theorem

$\oint P\,dx + Q\,dy = \iint(Q_x - P_y)\,dA$.

Self-check quiz

1. F = ∇φ means F is...

rotational

conservative

zero

constant

conservative. A gradient field.

2. Work of a conservative field around a closed loop:

2π

0

its area

undefined

0. φ(A) − φ(A).

3. Green’s theorem links a line integral to...

a derivative

a double integral

a triple integral

a series

a double integral. Boundary ↔ interior.

4. Q_x − P_y for F = ⟨−y, x⟩ is...

0

1

2

−2

2. 1 − (−1).

Try this quiz again

How this helps save the planet

Ocean gyres are huge rotating current systems: there are five main ones, and each collects floating debris toward its center, forming the garbage patches. The circulation of a current field is a line integral, and Green's theorem ties it to the rotation (curl) inside.

Winds and currents on maps such as earth.nullschool.net, which draws public NOAA forecast data, are vector fields you can read with this unit.

On Saving Planets: Oceans hub

Sources: NOAA Marine Debris Program: Garbage patches (https://marinedebris.noaa.gov/discover-marine-debris/garbage-patches) · NOAA Ocean Service: Ocean currents tutorial (gyres) (https://oceanservice.noaa.gov/education/tutorial_currents/04currents4.html) · earth.nullschool.net: live global wind and ocean-current map (from NOAA GFS and other public data) (https://earth.nullschool.net/)

Free learning resources

- OpenStax Calculus 3, 6.1 Vector Fields (https://openstax.org/books/calculus-volume-3/pages/6-1-vector-fields): Fields.
- OpenStax Calculus 3, 6.2 Line Integrals (https://openstax.org/books/calculus-volume-3/pages/6-2-line-integrals): Work and circulation.
- OpenStax Calculus 3, 6.3 Conservative Vector Fields (https://openstax.org/books/calculus-volume-3/pages/6-3-conservative-vector-fields): Potentials.
- OpenStax Calculus 3, 6.4 Green's Theorem (https://openstax.org/books/calculus-volume-3/pages/6-4-greens-theorem): Green's theorem.

Watch the lecture

Session 56 Clip: Vector Fields (https://ocw.mit.edu/courses/18-02sc-multivariable-calculus-fall-2010/resources/clip-vector-fields/) (in English)

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☐ **I finished this unit**

Surface Integrals, Stokes' and the Divergence Theorem

Divergence and curl, parametrized surfaces, flux through a surface, and the two great theorems that tie boundaries to interiors in 3D.

Learning objectives

By the end of this unit you can:

- Compute divergence and curl and interpret them as source strength and rotation.
- Parametrize surfaces and compute surface area and flux.
- Apply Stokes' theorem.
- Apply the Divergence (Gauss) theorem to net outward flux.

Key concepts

- div F = ∇ · F measures outflow per volume; curl F = ∇ × F measures rotation.
- Flux ∬_S F · n dS is how much of the field crosses a surface per unit time.
- Stokes: circulation around a surface's edge equals the flux of curl F through the surface.
- Divergence theorem: net outward flux through a closed surface equals the total divergence inside.

Key equations

$$\nabla \cdot \mathbf{F} = P_x + Q_y + R_z$$

· F=P_x+Q_y+R_z

Divergence

$$\nabla \times \mathbf{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$$

× F= R_y-Q_z,\ P_z-R_x,\ Q_x-P_y

Curl

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

_ S F· d r=_S(× F)· n dS

Stokes' theorem

$$\oiint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \nabla \cdot \mathbf{F}\, dV$$

_S F· n dS=_E· F dV

Worked examples, step by step

Example 8.1 Find the outward flux of $F = \langle x, y, z \rangle$ through the sphere of radius R.

1. div $F = 3$.
2. Divergence theorem: $3 \times (4\pi R^3/3)$.
3. Check directly: on the sphere $F \cdot n = R$, so the flux is $R \times 4\pi R^2$.

Answer: $4\pi R^3$ both ways.

Example 8.2 Sunlight of 1,361 W/m² falls on Earth (radius 6,371 km). What total power does Earth intercept, and what is the average per square meter of its surface?

1. The flux through Earth's shadow disk is $1{,}361 \times \pi R^2 = 1.735 \times 10^{17}$ W.
2. Spread over the whole sphere (area $4\pi R^2$) it is 1,361/4.

Answer: 1.74×10^{17} W, or 340 W/m² on average, the figure at the top of NASA's energy-budget diagram.

Example 8.3 Verify Stokes' theorem for $F = \langle -y, x, 0 \rangle$ on the upper half of the unit sphere.

1. curl $F = \langle 0, 0, 2 \rangle$.
2. Flux of curl through the hemisphere equals that through the flat unit disk (same edge): $2 \times \pi$.
3. The edge circulation is 2π (Unit 7).

Answer: 2π both ways.

Practice problems

Work each on paper first, then check.

1. Find div and curl of $F = \langle x^2, yz, xy \rangle$.

   div $= 2x + z$; curl $= \langle x - y, -y, 0 \rangle$.

2. Flux of $F = \langle 0, 0, 5 \rangle$ upward through the unit square in the plane $z = 1$.

   $5 \times 1 = 5$.

3. Use the Divergence theorem: flux of $F = \langle x, y, z \rangle$ out of the unit cube $[0, 1]^3$.

div = 3; flux 3.

4. Find the surface area of a sphere of radius 2 by integrating $dS = 4 \sin \varphi \, d\varphi \, d\theta$.

   $4 \times 2 \times 2\pi = 16\pi \approx 50.3$.

5. Is the curl of a gradient field ever nonzero?

   No: $\nabla \times \nabla\varphi = 0$ for smooth φ.

Common misconceptions

Myth: Flux is the same as circulation.

Actually: Flux crosses a surface; circulation runs along a curve.

Myth: Divergence-free means nothing moves.

Actually: It means no net source: whatever flows in, flows out.

Myth: The Divergence theorem works on open surfaces.

Actually: It needs a closed surface; Stokes handles surfaces with an edge.

Flashcards

Divergence

$\nabla \cdot F$; source strength.

Curl

$\nabla \times F$; rotation.

Flux

$\iint F \cdot n \, dS$.

Stokes' theorem

Edge circulation = flux of curl.

Divergence theorem

Outward flux = $\iiint$ div F dV.

Closed surface

Encloses a solid; no edge.

Self-check quiz

1. div ⟨x, y, z⟩ =

0

1

3

x + y + z

3. 1 + 1 + 1.

2. Stokes' theorem relates edge circulation to...

flux of F

flux of curl F

div F

volume

flux of curl F. $\iint (\nabla \times F) \cdot n \, dS$.

3. The Divergence theorem needs...

a closed surface

a curve

a flat region

a conservative field

a closed surface. It encloses a solid.

4. Average sunlight per square meter at Earth's top of atmosphere is about...

1,361 W/m²

680 W/m²

340 W/m²

100 W/m²

340 W/m². Disk area πR^2 over sphere area $4\pi R^2$: one quarter.

Try this quiz again

How this helps save the planet

Earth's climate is a flux balance: sunlight crosses the top of the atmosphere at about 340 W/m² on average, and Earth must send the same energy back out to space. When greenhouse gases trap some outgoing heat, the imbalance is a net flux into the planet, mostly stored in the ocean.

Pollutant transport is the same mathematics: the Divergence theorem says the pollutant leaving a lake or air basin through its boundary equals what its sources inside release, minus what is removed.

On Saving Planets: Earth Observation hub

Sources: NASA Earth Observatory: Climate and Earth's Energy Budget (https://earthobservatory.nasa.gov/features/EnergyBalance) · NASA Earth fact sheet (https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html)

Free learning resources

- OpenStax Calculus 3, 6.5 Divergence and Curl (https://openstax.org/books/calculus-volume-3/pages/6-5-divergence-and-curl): Div and curl.
- OpenStax Calculus 3, 6.6 Surface Integrals (https://openstax.org/books/calculus-volume-3/pages/6-6-surface-integrals): Flux.
- OpenStax Calculus 3, 6.7 Stokes' Theorem (https://openstax.org/books/calculus-volume-3/pages/6-7-stokes-theorem): Stokes.
- OpenStax Calculus 3, 6.8 The Divergence Theorem (https://openstax.org/books/calculus-volume-3/pages/6-8-the-divergence-theorem): Gauss.

Watch the lecture

Session 95 Clip: Stokes' Theorem and Surface Independence (https://ocw.mit.edu/courses/18-02sc-multivariable-calculus-fall-2010/resources/clip-stokes-theorem-and-surface-independence/) (in English)

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☐ **I finished this unit**

Problem Sets

These practice problems live inside each unit. Work on paper first, then open the answer.

1. Vectors and the Geometry of Space

5 problems

2. Vector Functions and Motion in Space
   4 problems
3. Partial Derivatives
   4 problems
4. Gradients and Optimization
   5 problems
5. Double Integrals
   5 problems
6. Triple Integrals and Change of Variables
   4 problems
7. Vector Fields, Line Integrals and Green’s Theorem
   4 problems
8. Surface Integrals, Stokes’ and the Divergence Theorem
   5 problems

Computer labs

Four computer labs apply the units: divergence and curl of a wind field from station data, slope and water-flow direction from an elevation grid, the volume of a lake by a polar double integral, and a satellite orbit stepped through time.

Sample data is labeled as such; each lab links to the public data source to try next.

Safety reference: OpenStax Calculus Volume 3 (free textbook) (https://openstax.org/books/calculus-volume-3/pages/1-introduction)

Lab A: Divergence and curl of a wind field

WEEK 14

Goal: Estimate the divergence and curl of the wind at a point from four surrounding stations, and interpret them as rising air and storm rotation.

$$\nabla \cdot \mathbf{v} \approx \frac{u_E - u_W}{2\Delta} + \frac{v_N - v_S}{2\Delta}, \quad (\nabla \times \mathbf{v})_z \approx \frac{v_E - v_W}{2\Delta} - \frac{u_N - u_S}{2\Delta}$$

Centered differences

Procedure

1. Enter the eastward (u) and northward (v) wind at stations 100 km east, west, north and south of a center (Δ = 100 km = 10^5 m).
2. Compute the divergence and the vertical curl (vorticity).
3. Interpret the signs, then repeat with real winds read off earth.nullschool.net or NOAA forecast data.

Data table

Station	u (m/s, east)	v (m/s, north)
East	4	8
West	6	−6
North	−5	−3
South	7	−1

Sample values for practice, not a real weather map.

Sample calculation

1. Divergence = (4 − 6)/(2 × 10^5) + (−3 − (−1))/(2 × 10^5) = −2 × 10^{-5} s^{-1}: air converges, so it must rise, which favors clouds and rain.
2. Vorticity = (8 − (−6))/(2 × 10^5) − (−5 − 7)/(2 × 10^5) = 1.3 × 10^{-4} s^{-1}: positive, counterclockwise, like a Northern Hemisphere low.

> **Safety**
>
> - None beyond normal computer use.

Home or virtual alternative

- A spreadsheet is enough.

earth.nullschool.net: live global wind and ocean-current map (from NOAA GFS and other public data) (https://earth.nullschool.net/) · NOAA NCEI: Global Forecast System (GFS) (https://www.ncei.noaa.gov/products/weather-climate-models/global-forecast)

Planet link: Convergence and vorticity are what forecasters track to warn of heavy rain and storms. NOAA/NWS JetStream: Wind (https://www.weather.gov/jetstream/wind)

- [ ] **I completed this lab (wet or virtual)**

Lab B: Slope and water flow from an elevation grid

WEEK 7

Goal: Estimate the gradient of terrain from a grid of elevations and find which way rainwater runs.

$$\nabla h \approx \left\langle \frac{h_E - h_W}{2\Delta}, \frac{h_N - h_S}{2\Delta} \right\rangle$$

Gradient by centered differences

Procedure

1. Use elevations 100 m east, west, north and south of a point (Δ = 100 m).
2. Compute ∇h, its size (slope) and the slope angle arctan|∇h|.
3. The flow direction is −∇h. Then download a free USGS 3DEP elevation tile and repeat for a hillside near you.

Data table

Neighbor	Elevation (m)
Center	250
East	262
West	240
North	256
South	246

Sample values for practice.

Sample calculation

1. ∇h ≈ ⟨(262 − 240)/200, (256 − 246)/200⟩ = ⟨0.11, 0.05⟩.
2. |∇h| = 0.121, a slope angle of about 6.9°.
3. Water runs along −∇h, to the west-southwest; tracing these directions over a grid outlines a watershed.

> **Safety**
>
> - None beyond normal computer use.

Home or virtual alternative

- A spreadsheet is enough; GIS software is optional.

USGS 3D Elevation Program (3DEP) (https://www.usgs.gov/3d-elevation-program)

Planet link: Flow directions from elevation grids are the basis of flood and erosion maps. USGS 3D Elevation Program (3DEP) (https://www.usgs.gov/3d-elevation-program)

- [ ] **I completed this lab (wet or virtual)**

Lab C: The volume of a lake by a polar double integral

WEEK 9

Goal: Compute the volume of a bowl-shaped lake exactly and with a numerical sum, and see how the error shrinks.

$$V = \int_0^{2\pi} \int_0^{R} D\left(1 - \frac{r^2}{R^2}\right) r \, dr \, d\theta = \frac{\pi R^2 D}{2}$$

V=_0^2π\!\!_0^RD(1-(r^2)/(R^2))r dr dθ=(π R^2D)/(2)

Lake with a paraboloid basin

Procedure

1. Take R = 1,000 m and maximum depth D = 20 m.
2. Compute the exact volume.
3. Split the lake into n rings, evaluate the depth at each ring's middle radius, and add depth × ring area. Try n = 5 and n = 10.

Data table

Method	Volume (m^3)
Exact	3.14×10^7
Midpoint, 5 rings	3.20×10^7
Midpoint, 10 rings	3.16×10^7

Computed values for a model lake.

Sample calculation

1. Exact: $\pi(1{,}000)^2(20)/2 = 3.14 \times 10^7$ m^3 (31.4 million cubic meters).
2. The 5-ring sum is 2.0% high and the 10-ring sum 0.5% high: doubling n cuts the error about four times.

> **Safety**
>
> - None beyond normal computer use.

Home or virtual alternative

- A spreadsheet or Python.

OpenStax Calculus 3, 5.3 Double Integrals in Polar Coordinates (https://openstax.org/books/calculus-volume-3/pages/5-3-double-integrals-in-polar-coordinates)

Planet link: Lake and reservoir volumes from depth surveys tell water managers how much they can store and release. NASA: SWOT mission (https://swot.jpl.nasa.gov/)

☐ **I completed this lab (wet or virtual)**

Lab D: Stepping a satellite orbit through time

WEEK 3

Goal: Integrate r'' = −GM r/|r|³ numerically for a Landsat-like orbit and compare two stepping methods.

$$\mathbf{v}_{n+1} = \mathbf{v}_n - \frac{GM\,\mathbf{r}_n}{|\mathbf{r}_n|^3}\Delta t, \quad \mathbf{r}_{n+1} = \mathbf{r}_n + \mathbf{v}_{n+1}\Delta t$$

```
v_n+1= v_n-(GM r_n)/(| r_n|^3)Δ t, r_n+1= r_n+ v_n+1Δ t
```

Semi-implicit (symplectic) Euler

Procedure

1. Start at r = ⟨7,076 km, 0⟩ with v = ⟨0, 7,505 m/s⟩ (Unit 2).
2. Step 592 times with Δt = 10 s (one period), first with plain Euler (update r with the old v), then with the semi-implicit version above.
3. Compare the final radius with the start.

Data table

Method (Δt = 10 s)	Radius change after one orbit
Plain Euler	+908 km
Semi-implicit Euler	+0.16 km

Computed values; GM = 3.986 × 10¹⁴ m³/s².

Sample calculation

1. Plain Euler spirals outward by about 908 km per orbit, a useless forecast; even Δt = 1 s still drifts about 94 km.
2. The semi-implicit update changes one line of code and keeps the orbit to within about 0.16 km.

> **Safety**
>
> - None beyond normal computer use.

Home or virtual alternative

- A spreadsheet works; Python in Colab is easier.

Google Colab (free notebooks) (https://colab.research.google.com/) · NASA: Landsat 9 quick facts (https://landsat.gsfc.nasa.gov/satellites/landsat-9/)

Planet link: Accurate orbit propagation keeps Earth-observing satellites on their repeat tracks.
NASA: Landsat 9 quick facts (https://landsat.gsfc.nasa.gov/satellites/landsat-9/)

☐ **I completed this lab (wet or virtual)**

Exam prep practice sets

Time yourself: about 10 minutes per problem. Then compare with the full solutions.

Midterm practice set (Units 1–4)

1. Find the angle between $\langle 1, 1, 0\rangle$ and $\langle 0, 1, 1\rangle$.

    1. Dot product 1; both lengths $\sqrt{2}$.
    2. $\cos\theta = 1/2$.

    Answer: 60°.

2. Find the plane through $(1, 1, 1)$ with normal $\langle 2, -1, 3\rangle$.

    1. $2(x - 1) - (y - 1) + 3(z - 1) = 0$.

    Answer: $2x - y + 3z = 4$.

3. A satellite orbits at radius 6,771 km (400 km altitude). Find its period.

    1. $T = 2\pi\sqrt{(r^3/GM)}$.

    Answer: ≈ 5,544 s ≈ 92.4 min.

4. Maximize $f = xy$ subject to $x + y = 10$.

    1. $\langle y, x\rangle = \lambda\langle 1, 1\rangle$ gives $x = y = 5$.

    Answer: 25 at (5, 5).

Final practice set (all units)

1. Evaluate $\int_0^{(2\pi)}\int_0^3 r\, dr\, d\theta$.

    1. $4.5 \times 2\pi$.

    Answer: $9\pi \approx 28.3$ (area of a disk of radius 3).

2. Find the volume of a hemisphere of radius 2 with spherical coordinates.

    1. $\int_0^{(2\pi)}\int_0^{(\pi/2)}\int_0^2 \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = (8/3)(1)(2\pi)$.

Answer: 16π/3 ≈ 16.76.

3. Find ∮ F · dr for F = ⟨−y, x⟩ around the square [0, 2] × [0, 2], counterclockwise.

    1. Green: ∬2 dA = 2 × 4.

    Answer: 8.

4. Find the outward flux of F = ⟨2x, 0, 0⟩ through the sphere of radius 1.

    1. div F = 2; flux = 2 × 4π/3.

    Answer: 8π/3 ≈ 8.38.

Formula sheet

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \qquad |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

$$n\cdot(\mathbf{r}-\mathbf{r}_0) = 0 \qquad L = \int |\mathbf{r}'(t)|\,dt$$

$$\kappa = |\mathbf{r}'\times\mathbf{r}''|/|\mathbf{r}'|^3 \qquad v = \sqrt{GM/r}$$

$$\Delta f \approx f_x\Delta x + f_y\Delta y \qquad D_{\mathbf{u}}f = \nabla f\cdot\mathbf{u}$$

$$D = f_{xx}f_{yy} - f_{xy}^2 \qquad \nabla f = \lambda\nabla g$$

$$dA = r\,dr\,d\theta \qquad dV = r\,dz\,dr\,d\theta$$

$$dV = \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \qquad \int_C \nabla\varphi\cdot d\mathbf{r} = \varphi(B)-\varphi(A)$$

$$\oint P\,dx + Q\,dy = \iint (Q_x - P_y)\,dA \qquad \oint \mathbf{F}\cdot d\mathbf{r} = \iint (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$$

$$\oiint \mathbf{F}\cdot\mathbf{n}\,dS = \iiint \nabla\cdot\mathbf{F}\,dV$$

Constants

Constant	Value
Earth’s GM	3.986×10^{14} m^3/s^2
Earth’s mean radius	6,371 km
Solar irradiance at Earth	1,361 W/m^2
Mean surface pressure	1,014 mb = 101,400 Pa
Atmospheric scale height	8.5 km
Standard gravity	9.81 m/s^2

Source: NASA Earth fact sheet (https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html)

Graphing and data tools

Use these free tools and data alongside the course.

- Desmos 3D calculator (https://www.desmos.com/3d): Graph surfaces, vectors and parametric curves in 3D.
- GeoGebra 3D (https://www.geogebra.org/3d): Another free 3D grapher.
- PhET: Vector Addition (https://phet.colorado.edu/en/simulations/vector-addition): Build intuition for components and sums.
- MIT OpenCourseWare 18.02SC (https://ocw.mit.edu/courses/18-02sc-multivariable-calculus-fall-2010/): Free MIT lectures, recitations and problem sets.
- earth.nullschool.net: live global wind and ocean-current map (from NOAA GFS and other public data) (https://earth.nullschool.net/): See real wind and current vector fields worldwide.
- USGS 3D Elevation Program (3DEP) (https://www.usgs.gov/3d-elevation-program): Free U.S. elevation data for gradient and slope work.

Glossary

Circulation
Line integral of a field around a closed curve.

Conservative field
A gradient field; work is path-independent.

Cross product
Vector perpendicular to two vectors.

Curl
$\nabla \times F$; local rotation of a field.

Curvature
How sharply a curve bends; 1/R for a circle.

Directional derivative
Rate of change in a given direction, $\nabla f \cdot u$.

Divergence
$\nabla \cdot F$; net outflow per volume.

Divergence theorem
Outward flux through a closed surface equals the integral of the divergence inside.

Dot product
$a \cdot b = |a||b| \cos \theta$.

Flux
Amount of a field crossing a surface.

Gradient
Vector of partial derivatives; points uphill.

Green's theorem
Circulation around a plane curve equals the integral of $Q_x - P_y$ inside.

Jacobian
Stretch factor of a change of variables.

Lagrange multiplier
λ in $\nabla f = \lambda \nabla g$ for constrained optimization.

Level curve
Curve where $f(x, y)$ is constant.

Line integral
Integral along a curve.

Partial derivative
Derivative in one variable with the others fixed.

Saddle point
Critical point that is neither a max nor a min.

Stokes' theorem
Edge circulation equals the flux of the curl.

Vector field
Assignment of a vector to each point.

Self-assessment rubric

No grade or credit is given. Use this to judge honestly where you are in each unit.

Level	What it looks like
4 · Mastery	Sets up integrals and fields from a picture unprompted, picks good coordinates, and checks answers by a second method or a theorem.

Level	What it looks like
3 · Proficient	Solves the practice problems with occasional slips.
2 · Developing	Follows worked examples but struggles to set up limits or choose coordinates.
1 · Beginning	Applies formulas without sketching or checking; needs to revisit the unit.

Suggested weighting if you want a score

Part	Weight
Practice problems	30%
Computer labs	15%
Midterm practice set	25%
Final practice set	30%

Aim for level 3 or better in every unit before moving to the next course.

Related courses

- Intro Programming ENGR 130
- Differential Equations MATH 220
- Physics II PHYS 210 + PHYS 210L
- All courses